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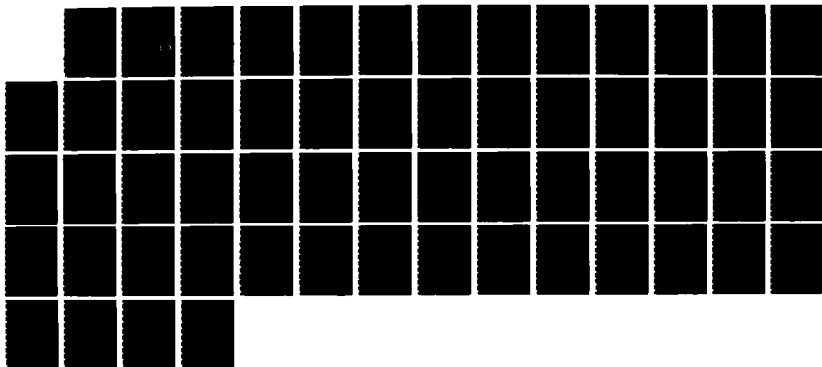
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CAROLINA UNIV AT CHARLOTTE DEPT OF MATHEMATICS
R F ANDERSON 1986 AFOSR-TR-86-2144 AFOSR-80-8245

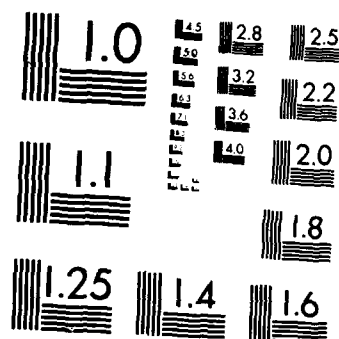
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REPLACEMENT WITH NON-CONSTANT OPERATING COST

R. F. Anderson†*

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SECURITY CLASSIFICATION OF THIS PAGE

REPORT DOCUMENTATION PAGE

1a. REPORT SECURITY CLASSIFICATION Unclassified			1b. RESTRICTIVE MARKINGS	
2a. SECURITY CLASSIFICATION AUTHORITY			3. DISTRIBUTION/AVAILABILITY OF REPORT Approved for public release; distribution unlimited.	
2b. DECLASSIFICATION/DOWNGRADING SCHEDULE				
4. PERFORMING ORGANIZATION REPORT NUMBER(S)			5. MONITORING ORGANIZATION REPORT NUMBER(S) AFOSR-TR. 86-2144	
6a. NAME OF PERFORMING ORGANIZATION University of North Carolina		6b. OFFICE SYMBOL (If applicable) nm		7a. NAME OF MONITORING ORGANIZATION AFOSR/NM
6c. ADDRESS (City, State and ZIP Code) University of North Carolina at Charlotte Charlotte, North Carolina 28223			7b. ADDRESS (City, State and ZIP Code) Bldg. 410 Bolling AFB, DC 20332-6448	
8a. NAME OF FUNDING/SPONSORING ORGANIZATION AFOSR		8b. OFFICE SYMBOL (If applicable) nm		9. PROCUREMENT INSTRUMENT IDENTIFICATION NUMBER AFOSR-80-0245
8c. ADDRESS (City, State and ZIP Code) Bolling Air Force Base Washington, DC 20332-6448			10. SOURCE OF FUNDING NOS.	
			PROGRAM ELEMENT NO. 61102F	PROJECT NO. 2304
11. TITLE (Include Security Classification) "Replacement with Non-Constant Operating Cost"				
12. PERSONAL AUTHOR(S) R.F. Anderson				
13a. TYPE OF REPORT reprint		13b. TIME COVERED FROM _____ TO _____		14. DATE OF REPORT (Yr., Mo., Day)
				15. PAGE COUNT 56 (fifty-six)
16. SUPPLEMENTARY NOTATION				
17. COSATI CODES			18. SUBJECT TERMS (Continue on reverse if necessary and identify by block number)	
FIELD	GROUP	SUB. GR.		
19. ABSTRACT (Continue on reverse if necessary and identify by block number) The long run average cost problem is considered in the case of a non-decreasing Markov wear process with failure determined by a random threshold. The method of analysis is to first consider the discounted problem and then let the discount factor go to zero.				
20. DISTRIBUTION/AVAILABILITY OF ABSTRACT UNCLASSIFIED/UNLIMITED <input checked="" type="checkbox"/> SAME AS RPT. <input type="checkbox"/> DTIC USERS <input type="checkbox"/>			21. ABSTRACT SECURITY CLASSIFICATION Unclassified	
22a. NAME OF RESPONSIBLE INDIVIDUAL Capt Thomas			22b. TELEPHONE NUMBER (Include Area Code) 767-5025	22c. OFFICE SYMBOL nm

REPLACEMENT WITH NON-CONSTANT OPERATING COST

R. F. Anderson†*

ABSTRACT

The long run average cost problem is considered in the case of a non-decreasing Markov wear process with failure determined by a random threshold. The method of analysis is to first consider the discounted problem and then let the discount factor go to zero.

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* This work was partially supported by AFOSR 80-0245



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§1. Introduction.

Assume that one has a machine whose failure is described by a wear process $x_t: t \geq 0$ which is a positive non-decreasing Markov process and a random threshold $Y \geq 0$ independent of $x_t: t \geq 0$ with failure occurring at time $\sigma = \inf\{t: x_t \geq Y\}$. At any time one can replace the machine by a new one with the same mode of operation. There is an operating cost $f(x)$ per unit of time and a replacement cost $g(x)$ if replacement is done before failure and replacement cost c_0 if replacement is done at failure. Note that replacement is always to replace the machine by a new one. Models of this type have been considered in the reliability literature by Abdel-Hameed [1], [2], Drosen [5], and Taylor [10].

The stochastic control problem of minimizing the cost is generally called the Optimal Replacement Problem and has been considered by the above authors in the case that f is constant. The interest has been in the long run average cost problem. In this work we will consider the case of general $f(x)$. We first view the problem as a discounted cost problem, and as in Robin [8], we obtain the long run average cost by letting the discount factor go to zero. The main difference with the work of Robin is that the invariant measure is not obtained exponentially fast by the Markov transition probabilities of the replacement process as time goes to infinity. (See §4).

In Section 2 we state the Long Run Average Cost Optimal Replacement Problem and deal with a preliminary discounted optimal stopping problem. Section 3 formulates and solves the discounted optimal replacement problem. The replacement process is introduced in Section 4, the invariant measure is found, and ergodic results are derived for the linear problem. Section 5 proves the main technical result and Section 6 contains the main result that the solution to the discounted problem suitably modified converges to the solution of the long run average problem.

§2. Notation, Statement of the Problem, and a Preliminary Stopping Problem.

Let $\Omega = D(R^+, R^+)$ be the space of right continuous functions with left limits. Here $R^+ = [0, \infty)$. Let $x_t(\omega) = \omega(t)$ for $\omega \in \Omega$, $F_t^0 = \sigma(x_s : 0 \leq s \leq t)$, $F^0 = F_\infty^0$, and F_t, F the universally completed σ -fields F_t^0 and F^0 respectively. Let $(\Omega, F_t, x_t : t \geq 0, P_x)$ be a homogeneous, non-decreasing, non-negative Markov process with associated semi-group $T_t : t \geq 0$ defined on $C_b(R^+)$, the space of bounded real valued continuous functions defined on R^+ with norm taken to be supremum norm. We assume

$$(2.1) \quad T_t : t \geq 0 \text{ is Feller, that is, for } f \in C_b(R^+) \\ T_t f \in C_b(R^+) \text{ and } T_t f \rightarrow f \text{ in supremum norm as } t \rightarrow 0.$$

Let A denote the infinitesimal generator of $T_t : t \geq 0$ and D_A its domain in $C_b(R^+)$. Assume also

$$(2.2) \quad x_t : t \geq 0 \text{ is quasi-left continuous, that is if } \tau_n : n \geq 1 \text{ is a} \\ \text{sequence of stopping times with } \tau_n \uparrow \tau, \text{ then } x_{\tau_n} \rightarrow x_\tau \text{ a.s. } P_x \\ \text{on the set } (\tau < \infty). \text{ See Dynkin [6], Vol. 1 pp. 103.}$$

Let Y be a positive random variable independent of $x_t : t \geq 0$ with a continuous distribution function $G(y)$. Let $\bar{G}(y) = 1 - G(y)$. Define

$$\sigma = \inf\{t : x_t \geq Y\}$$

and let $H(t) = P_0(\sigma \leq t) = P_0(x_t \geq Y)$. Assume

$$(2.3) \quad E_0[\sigma] < \infty \quad \text{and} \quad H(0) = 0.$$

Let

$$(2.4) \quad f, g \in C_b(R^+) \quad f, g \geq 0 \quad \text{and} \quad c_0 > 0 \quad \text{a constant.}$$

We say $\tau : \Omega \rightarrow [0, \infty]$ is a stopping time with respect to $x_t : t \geq 0$ if $(\tau \leq t) \in F_t$ for all t .

Let

$$\lambda = \inf_{\tau} \frac{E_0 \left[\int_0^{\sigma \wedge \tau} f(x_s) ds + I_{(\tau < \sigma)} g(x_{\tau}) + I_{(\tau \geq \sigma)} c_0 \right]}{E_0[\sigma \wedge \tau]}$$

The Long Run Average Optimal Replacement Problem is to find $\hat{\tau}$ so that

$$\lambda = \frac{E_0 \left[\int_0^{\sigma \wedge \hat{\tau}} f(x_s) ds + I_{(\hat{\tau} < \sigma)} g(x_{\hat{\tau}}) + I_{(\hat{\tau} \geq \sigma)} c_0 \right]}{E_0[\sigma \wedge \hat{\tau}]}$$

Our first step is to establish a result for a discounted optimal stopping problem. For the following see Robin [9] or Bensoussan [3]:

Lemma 2.1. For $b \geq 0$ fixed and $\alpha > 0$, let $V_b \in C_b(R^+)$ be the maximal solution of

$$U \in C_b(R^+) \quad U(x) \leq b + g(x) \bar{G}(x) + c_0 G(x)$$

$$U \leq e^{-\alpha t} T_t U + \int_0^t e^{-\alpha s} T_s (f \bar{G} + \alpha(b + c_0)G) ds$$

then

$$V_b(x) = \inf_{\tau} J_x^b(\tau)$$

where

$$J_x^b(\tau) = E_x \left[\int_0^{\tau} e^{-\alpha s} (f(x_s) \bar{G}(x_s) + \alpha(b + c_0)G(x_s)) ds + e^{-\alpha \tau} (b + g(x_{\tau}) \bar{G}(x_{\tau}) + c_0 G(x_{\tau})) \right]$$

Moreover if

$$\hat{\tau}_b = \inf\{t: V_b(x_t) = b + g(x_t) \bar{G}(x_t) + c_0 G(x_t)\}.$$

then

$$V_b(x) = J_x^b(\hat{\tau}_b).$$

Lemma 2.2. (i) For $0 \leq b_0 < b_1$, $V_{b_0}(x) \leq V_{b_1}(x)$

$$(ii) |V_{b_0}(x) - V_{b_1}(x)| \leq 2|b_0 - b_1|$$

Proof. (i) If $0 \leq b_0 < b_1$, then $J_x^{b_0}(\tau) \leq J_x^{b_1}(\tau)$ for any stopping time τ .

Hence $V_{b_0}(x) \leq V_{b_1}(x)$.

(ii) For any τ ,

$$J_x^{b_0}(\tau) - J_x^{b_1}(\tau) = E_x \left[\int_0^\tau \alpha e^{-\alpha s} (b_0 - b_1) G(x_s) ds + e^{-\alpha \tau} (b_0 - b_1) \bar{G}(x_\tau) \right]$$

Since G is a distribution function,

$$|J_x^{b_0}(\tau) - J_x^{b_1}(\tau)| \leq 2|b_0 - b_1| \quad \text{and therefore}$$

$$|V_{b_0}(x) - V_{b_1}(x)| \leq 2|b_0 - b_1|.$$

Lemma 2.3. If $H(0) = 0$, there exists a $b_0 > 0$ so that

$$V_{b_0}(0) < b_0.$$

For such a b_0 , define inductively

$$b_k = V_{b_{k-1}}(0) \quad k \geq 1$$

Then $b_k \uparrow \bar{b}$ where $\bar{b} \geq 0$. Moreover $V_{b_k}(x) \uparrow V_{\bar{b}}(x)$ in $C_b(R^+)$ and $V_{\bar{b}}(0) = \bar{b}$.

Proof. For any $b \geq 0$ take $\tau \equiv 0$ to obtain

$$V_b(0) \leq E_0 \left[\int_0^\infty e^{-\alpha s} (f(x_s) \bar{G}(x_s) + \alpha(b + c_0) G(x_s)) ds \right]$$

Note that

$$H(t) = P_0(\sigma \leq t) = P_0(x_t \geq Y) = E_0[G(x_t)]$$

and so

$$\hat{H}(\alpha) = E_0[e^{-\alpha \sigma}] = E_0 \left[\int_0^\infty \alpha e^{-\alpha s} G(x_s) ds \right]$$

Since \hat{H} is not point mass at 0, $H(\alpha) < 1$.

Let

$$z_0 = E_0 \left[\int_0^\infty e^{-\alpha s} (f(x_s) \bar{G}(x_s) + \alpha c_0 G(x_s)) ds \right]$$

and we have

$$V_b(0) \leq z_0 + b \hat{H}(\alpha).$$

Select b_0 large enough so that

$$z_0 + b_0 \hat{H}(\alpha) < b_0$$

Next for such a b_0 , we have

$$b_1 = V_{b_0}(0) < b_0$$

and by Lemma 2.2, it follows inductively that

$$b_k = V_{b_{k-1}}(0) \leq V_{b_{k-1}}(0) = b_{k-1}$$

Define $\bar{b} = \lim b_k$, and let $V_{\bar{b}}(x)$ be given by Lemma 2.1.

From Lemma 2.2,

$$|V_{b_k}(x) - V_{\bar{b}}(x)| \leq 2|b_k - \bar{b}|$$

and so $V_{b_k} \rightarrow V_{\bar{b}}$ in $C_b(\mathbb{R}^+)$ and moreover

$$\bar{b} = \lim_k b_k = \lim_k V_{b_{k-1}}(0) = V_{\bar{b}}(0).$$

We thus have established the following.

Theorem 2.4. Under the assumption (2.1) - (2.4), let $V^\alpha(x) = V_{\bar{b}}(x)$, \bar{b} as in Lemma 2.3. Then

$$V^\alpha(x) = \inf_{\tau} J_x^\alpha(\tau)$$

where

$$J_x^\alpha(\tau) = E_x \left[\int_0^\tau e^{-\alpha s} (f(x_s) \bar{G}(x_s) + \alpha (V^\alpha(0) + c_0) G(x_s)) ds \right. \\ \left. + e^{-\alpha \tau} (V^\alpha(0) + g(x_\tau) \bar{G}(x_\tau) + c_0 G(x_\tau)) \right]$$

Moreover if

$$\hat{\tau} = \inf \{t: V^\alpha(x_t) = V^\alpha(0) + g(x_t)\bar{G}(x_t) + c_0 G(x_t)\}$$

then

$$V^\alpha(x) = J_x^\alpha(\hat{\tau}).$$

Remark 2.1. It will be shown in Section 3 that $V^\alpha(x)$ is the maximal solution of a quasi-variational inequality.

Corollary 2.5. Under the assumption $g(0) > 0$ and these of Theorem 2.4,

$$\hat{\tau} > 0 \text{ a.s. } P_0$$

Proof. If $g(0) > 0$ then taking $\tau \equiv 0$

$$V^\alpha(0) < V^\alpha(0) + g(0)$$

and hence there is a $\delta > 0$ so that if $0 \leq x < \delta$ then

$$V^\alpha(x) + \varepsilon < V^\alpha(0) + g(x)\bar{G}(x) + c_0 G(x)$$

for $\varepsilon > 0$ sufficiently small. Hence

$$\hat{\tau} > \tau_{\delta/2} = \inf \{t: x(t) \geq \delta/2\}$$

and since $x_t: t \geq 0$ is right continuous, $\tau_{\delta/2} > 0$ a.s. P_0 .

§3. The Discounted Optimal Replacement Problem.

Let $x_t^k: t \geq 0$ and Y^k $k \geq 1$ be independent copies of $x_t: t \geq 0$ and Y .

Define

$$\sigma_k = \inf \{t: x_t^k \geq Y^k\}$$

Let f, g and c_0 satisfy (2.4). Suppose $\tau_k: k \geq 1$ are stopping times respectively with respect to $F_t^k: t \geq 0$ the universal completion of $\sigma(x_s^k: s \leq t)$ and assume for all n $\sum_{k \geq n} \tau_k = \infty$ a.s. P_0 . We use the notation

$\vec{\tau} = (\tau_1, \tau_2, \dots)$ and refer to $\vec{\tau}$ as replacement times. Define

$$\begin{aligned} \tilde{J}_0^\alpha(\vec{\tau}) = & E_0 \left[\int_0^{\sigma_1 \wedge \tau_1} e^{-\alpha s} f(x_s^1) ds + e^{-\alpha \sigma_1 \wedge \tau_1} (g(x_{\tau_1}^1) I_{(\tau_1 < \sigma_1)} + c_0 I_{(\tau_1 \geq \sigma_1)}) \right. \\ & + \sum_{n=2}^{\infty} \prod_{\ell=1}^{n-1} e^{-\alpha \sigma_\ell \wedge \tau_\ell} \left(\int_0^{\sigma_n \wedge \tau_n} e^{-\alpha s} f(x_s^n) ds \right. \\ & \left. \left. + e^{-\alpha \sigma_n \wedge \tau_n} (g(x_{\tau_n}^n) I_{(\tau_n < \sigma_n)} + c_0 I_{(\tau_n \geq \sigma_n)}) \right) \right]. \end{aligned}$$

Here it is assumed that $P_0(x_0^k = 0) = 1$ for all k . Let

$$V_0^\alpha = \inf_{\vec{\tau}} \tilde{J}_0^\alpha(\vec{\tau})$$

We seek $\hat{\vec{\tau}}$ so that

$$V_0^\alpha = \tilde{J}_0^\alpha(\hat{\vec{\tau}})$$

This we refer to as the Discounted Optimal Replacement Problem.

Define

$$\begin{aligned} J_0(\tau_1) = & E_0 \left[\int_0^{\sigma_1 \wedge \tau_1} e^{-\alpha s} f(x_s^1) ds + e^{-\alpha \sigma_1 \wedge \tau_1} (V_0^\alpha + g(x_{\tau_1}^1) I_{(\tau_1 < \sigma_1)} \right. \\ & \left. + c_0 I_{(\tau_1 \geq \sigma_1)}) \right]. \end{aligned}$$

Lemma 3.1. $V_0^\alpha = \inf_{\tau_1} J_0^\alpha(\tau_1)$

Proof. Let $\tau = (\tau_1, \tau_2, \dots)$ be any replacement sequence.

By independence letting $\hat{\tau} = (\tau_2, \tau_3, \dots)$

$$\begin{aligned} \tilde{J}_0^\alpha(\tau) = E_0 \left[\int_0^{\sigma_1 \wedge \tau_1} e^{-\alpha s} f(x_s^1) ds + e^{-\alpha \sigma_1 \wedge \tau_1} (\tilde{J}_0^\alpha(\hat{\tau}) + g(x_{\tau_1}^1) I_{(\tau_1 < \sigma_1)} \right. \\ \left. + c_0 I_{(\tau_1 \geq \sigma_1)}) \right] \geq J_0^\alpha(\tau_1) \end{aligned}$$

Therefore

$$V_0^\alpha \geq \inf_{\tau_1} J_0^\alpha(\tau_1) \equiv d$$

For $\varepsilon > 0$ choose τ_1 and $\hat{\tau}$ so that

$$J_0^\alpha(\tau_1) \leq d + \varepsilon \quad \text{and} \quad \tilde{J}_0^\alpha(\hat{\tau}) \leq V_0^\alpha + \varepsilon$$

Let $\vec{\tau}_0 = (\tau_1, \tau)$ then

$$V_0^\alpha \leq \tilde{J}_0^\alpha(\vec{\tau}_0) \leq J_0^\alpha(\tau_1) + \varepsilon \leq d + 2\varepsilon.$$

Since ε was arbitrary, we have equality.

In what follow $x_t: t \geq 0$ and Y will be generic copies of the wear process and the random threshold and $\sigma = \inf\{t: x_t \geq Y\}$. Note (2.3) insures $\sigma < \infty$ a.s. P_0 .

Lemma 3.2. for any stopping time τ with respect to $F_t: t \geq 0$ on the set $(\tau < \infty)$

$$P_0(\sigma \leq \tau | F_\tau) = P_0(x_\tau \geq Y | F_\tau) = G(x_\tau)$$

$$\text{and} \quad P_0(\sigma > \tau | F_\tau) = P_0(x_\tau < Y | F) = \bar{G}(x_\tau).$$

On the set $(\tau = \infty)$,

$$P_0(\sigma \leq \tau | F_\tau) = 1 \quad \text{and} \quad P_0(\sigma > \tau | F_\tau) = 0.$$

Proof. First note that by the independence of $x_t: t \geq 0$ and Y , and the assumption that $x_t: t \geq 0$ is pathwise non-decreasing, for any fixed t .

$$P_0(\sigma \leq t | F_t) = P_0(x_t \geq Y | F_t) = G(x_t)$$

For any τ an $F_t: t \geq 0$ stopping time, define

$$\tau_n = \begin{cases} k/n & \text{on } (k-1/n < \tau \leq k/n) \\ \infty & \text{on } (\tau = \infty) \end{cases} = A_{k/n}$$

Since $A_{k/n} \in F_{k/n}$, on the set $(\tau_n < \infty)$

$$\begin{aligned} P_0(\sigma \leq \tau_n | F_{\tau_n}) &= \sum_1^{\infty} P_0(\sigma \leq k/n | F_{k/n}) I_{A_{k/n}} \\ &= \sum_1^{\infty} G(x_{k/n}) I_{A_{k/n}} = G(x_{\tau_n}) \end{aligned}$$

Because $\tau \leq \tau_n$, $F_{\tau} \leq F_{\tau_n}$ and so on the set $(\tau < \infty)$

$$\begin{aligned} P_0(\sigma \leq \tau | F_{\tau}) &= P_0\left(\bigcap_1^{\infty} (\sigma \leq \tau_n) | F_{\tau}\right) \\ &= \lim_n E_0[P_0(\sigma \leq \tau_n | F_{\tau_n}) | F_{\tau}] \\ &= \lim_n E_0[G(x_{\tau_n}) | F_{\tau}] = G(x_{\tau}) \end{aligned}$$

The last follows since $\tau_n \downarrow \tau$ and $x_t: t \geq 0$ is right continuous.

On the set $(\tau = \infty)$ there is nothing to show.

Lemma 3.3. Let f and g be bounded and continuous and $\alpha > 0$. Then for any stopping time τ with respect to $F_t: t \geq 0$.

$$(i) \quad E_0 \left[\int_0^{\sigma \wedge \tau} e^{-\alpha s} f(x_s) ds \right] = E_0 \left[\int_0^{\tau} e^{-\alpha s} f(x_s) \bar{G}(x_s) ds \right]$$

$$(ii) \quad E_0 \left[e^{-\alpha \sigma \wedge \tau} g(x_\tau) I_{(\tau < \sigma)} \right] = E_0 \left[e^{-\alpha \tau} g(x_\tau) \bar{G}(x_\tau) \right]$$

$$(iii) \quad E_0 \left[e^{-\alpha \sigma \wedge \tau} I_{(\tau \geq \sigma)} \right] = E_0 \left[e^{-\alpha \tau} G(x_\tau) + \int_0^\tau \alpha e^{-\alpha s} G(x_s) ds \right]$$

Proof. (i) By Lemma 3.2

$$P_0(x_s < Y | F_s) = \bar{G}(x_s)$$

and so on the set $(s < \tau)$

$$P_0(x_s < Y | F_\tau) = \bar{G}(x_s)$$

Hence

$$\begin{aligned} E_0 \left[\int_0^{\tau \wedge \sigma} e^{-\alpha s} f(x_s) ds \right] &= E_0 \left[\int_0^\tau e^{-\alpha s} f(x_s) I_{(x_s < Y)} ds \right] \\ &= E_0 \left[\int_0^\tau e^{-\alpha s} f(x_s) \bar{G}(x_s) ds \right] \end{aligned}$$

(ii) This follows directly from Lemma 3.2.

(iii) Note that

$$e^{-\alpha \sigma \wedge \tau} I_{(\tau \geq \sigma)} = e^{-\alpha \tau} I_{(x_\tau \geq Y)} + \int_0^\tau \alpha e^{-\alpha s} I_{(x_s \geq Y)} ds I_{(x_\tau \geq Y)}.$$

Since $x_t: t \geq 0$ is pathwise non-decreasing

$$\int_0^\tau \alpha e^{-\alpha s} I_{(x_s \geq Y)} ds I_{(x_\tau \geq Y)} = \int_0^\tau \alpha e^{-\alpha s} I_{(x_s \geq Y)} ds$$

The rest is Lemma 3.2.

Lemma 3.4.
$$V_0^\alpha = \inf_\tau E_0 \left[\int_0^\tau e^{-\alpha s} (f(x_s) \bar{G}(x_s) + \alpha (V_0^\alpha + c_0) G(x_s)) ds \right. \\ \left. + e^{-\alpha \tau} (V_0^\alpha + g(x_\tau) G(x_\tau) + c_0 G(x_\tau)) \right].$$

Proof. This is immediate from Lemma 3.1 and Lemma 3.3.

Theorem 3.5. Suppose $g(0) > 0$ and (2.1)-(2.4) are satisfied. Let $V^\alpha(x)$ be

the maximal solution of $U(x) \leq U(0)$

$$(3.1) \quad U \in C_b(R^+) \quad U(x) \leq U(0) + g(x)\bar{G}(x) + c_0 G(x) \quad \text{and}$$

$$U(x) \leq e^{-\alpha t} T_t U(x) + \int_0^t e^{-\alpha s} T_s [f\bar{G} + \alpha(U(0) + c_0 G)](x) ds$$

then

$$(3.2) \quad V^\alpha(x) = \inf_{\tau} J_x^\alpha(\tau) \quad \text{and}$$

with $\hat{\tau} = \inf \{t: V^\alpha(x_t) = V^\alpha(0) + g(x_t)\bar{G}(x_t) + c_0 G(x_t)\}$

$$(3.3) \quad V^\alpha(x) = J_x^\alpha(\hat{\tau})$$

Moreover

$$(3.4) \quad V^\alpha(0) = \inf_{\tau} \tilde{J}_0^\alpha(\tau)$$

and if $\hat{\tau} = (\hat{\tau}_1, \hat{\tau}_2, \dots)$ where $\hat{\tau}_k$ is defined the same as $\hat{\tau}$ above except x_t^k replaces x_t , then

$$(3.5) \quad V^\alpha(0) = J_0^\alpha(\tau).$$

Proof. Let $V^\alpha(x)$ be as in Theorem 2.4. By Lemma 2.1, $V^\alpha(x)$ solves (3.1). Moreover (3.2) and (3.3) are consequences of Theorem 2.4. Also Lemmas 3.1 and 3.4 prove (3.4). What is left is to show (3.5) and $V^\alpha(x)$ is the maximal solution of (3.1).

Let $U(x)$ be any solution of (3.1) and τ be any stopping time. By the Markov property,

$$e^{-\alpha t} U(x_t) + \int_0^t e^{-\alpha s} f(x_s) \bar{G}(x_s) + \alpha(U(0) + c_0 G(x_s)) ds$$

is a submartingale. Hence

$$U(x) \leq E_x \left[e^{-\alpha t \wedge \tau} U(x_{t \wedge \tau}) + \int_0^{t \wedge \tau} e^{-\alpha s} (f(x_s) \bar{G}(x_s) + \alpha(U(0) + c_0 G(x_s))) ds \right]$$

By (2.2) letting $t \rightarrow \infty$,

$$(3.6) \quad U(x) \leq E_x \left[e^{-\alpha \tau} U(x) + \int_0^\tau e^{-\alpha s} (f(x_s) \bar{G}(x_s) + \alpha(U(0) + c_0)G(x_s)) ds \right] \\ \leq E_x \left[\int_0^\tau e^{-\alpha s} f(x_s) \bar{G}(x_s) + \alpha(U(0) + c_0)G(x_s) ds \right. \\ \left. + e^{-\alpha \tau} (U(0) + g(x_\tau) \bar{G}(x_\tau) + c_0 G(x_\tau)) \right]$$

If we can show $U(0) \leq V^\alpha(0)$ then

$$U(x) \leq J_\alpha^x(\tau) \quad \text{and so} \quad U(x) \leq V^\alpha(x)$$

By (3.6) and Lemma 3.3

$$(3.7) \quad U(0) \leq E_0 \left[\int_0^{\sigma \wedge \tau} e^{-\alpha s} f(x_s) ds + e^{-\alpha \sigma \wedge \tau} (U(0) + g(x_\tau) I_{(\tau < \sigma)} + c_0 I_{(\tau \geq \sigma)}) \right]$$

If $\vec{\tau} = (\tau_1, \tau_2, \dots)$ is any replacement sequence, then using (3.7) inductively, after n steps we have

$$(3.8) \quad U(0) \leq E_0 \left[\int_0^{\sigma_1 \wedge \tau_1} e^{-\alpha s} f(x_s^1) ds + e^{-\alpha \sigma_1 \wedge \tau_1} (g(x_{\tau_1}^1) I_{(\tau_1 < \sigma_1)} + c_0 I_{(\tau_1 \geq \sigma_1)}) \right. \\ + \sum_{j=2}^n \prod_{\ell=1}^{j-1} e^{-\alpha \sigma_\ell \wedge \tau_\ell} \int_0^{\sigma_j \wedge \tau_j} e^{-\alpha s} f(x_s^j) ds \\ \left. + e^{-\alpha \sigma_j \wedge \tau_j} (g(x_{\tau_j}^j) I_{(\tau_j < \sigma_j)} + c_0 I_{(\tau_j \geq \sigma_j)}) + \prod_{\ell=1}^n e^{-\alpha \sigma_\ell \wedge \tau_\ell} U(0) \right]$$

By (2.3) since the σ_ℓ 's are independent, and $\sum_{j \geq n} \tau_j = \infty$ for all n a.s. P_0 ,

we obtain by letting $n \rightarrow \infty$

$$U(0) \leq \tilde{J}_0^\alpha(\vec{\tau}), \text{ i.e. } U(0) \leq V_0^\alpha = V^\alpha(0).$$

To obtain (3.5), repeat the above argument using $V^\alpha(0)$ and $\hat{\tau}$. At each step we have equality and so (3.8) is an equality. Corollary 2.5 assures that $\sum_{\hat{\tau}_j = \infty} \text{a.s. } P_0$, and so $V^\alpha(0) = J_0^\alpha(\tau)$.

§4. The Replacement Process.

Let $x_t^k: t \geq 0$ and Y^k be as in Section 3 and as before $\sigma_k = \inf\{t: x_t^k \geq Y^k\}$. Assume that $P_0(x_0^k = 0 \text{ for all } k) = 1$ and define $\sigma_0 \equiv 0$. Let

$$(4.1) \quad z_t = x_t^k \text{ on the set } \sum_{\ell=0}^{k-1} \sigma_\ell \leq t < \sum_{\ell=0}^k \sigma_\ell.$$

We refer to $z_t: t \geq 0$ as the replacement process. Let $H(t)$ be as in (2.3).

For $B \subset \mathbb{R}^+$, Borel, by Lemma 3.2 and the independence of $x_t^k: t \geq 0$ and Y^k ,

$$\begin{aligned} P_0(z_t \in B) &= \sum_{k=1}^{\infty} P_0 \left(\sum_{\ell=0}^{k-1} \sigma_\ell \leq t, \quad x_{t-k-1}^k \in B, \quad x_{t-k-1}^k < Y^k \right) \\ &= \sum_{k=1}^{\infty} \int_0^t E_0 [I_B(x_{t-u}) \bar{G}(x_{t-u})] H^{(k-1)}(du) \\ &= \int_0^t \int_B p_{t-u}(0, dz) \bar{G}(z) R(du) \end{aligned}$$

where $H^{(k)}(t)$ is the k -fold convolution of $H(t)$, $R(t) = \sum_{k=0}^{\infty} H^{(k)}(t)$, and

$p_t(0, dz) = P_0(x_t \in dz)$. It is standard from renewal theory (see Feller [7]), that $R(t) < \infty$.

Suppose $G(x) < 1$. Considerations of x for which $G(x) = 1$ are not necessary because $z_t: t \geq 0$ never reaches x . Define Y_x by defining for $y > x$

$$P(Y_x \leq y) = P(Y \leq y | Y > x) = \frac{G(y) - G(x)}{\bar{G}(x)} = G_x(y)$$

We use the notation $P_{x,0}$ to stand for the condition

$P_{x,0}(x_0^1 = x, x_0^k = 0, k \geq 2) = 1$. Take Y_x^1 to have distribution $G_x(y)$ and maintain the usual independence.

Define

$$\sigma_x = \inf\{t: x_t^1 \geq Y_x^1\}.$$

Let $z_t: t \geq 0$ be as in (4.1) except that $\sigma_1 = \sigma_x$.

Define $H_x(t) = P_x(\sigma_x \leq t)$. By a slight modification of Lemma 3.2,

$$\begin{aligned} (4.2) \quad P_x(z_t \in B) &= P_{x,0}(x_t^1 \in B, x_t^1 \leq Y_x^1) \\ &+ \sum_{k=2}^{\infty} P_{x,0} \left(\sum_{\ell=0}^{k-1} \sigma_{\ell} \leq t, x_{t-k-1}^k \in B, x_{t-k-1}^k < Y_x^k \right) \\ &= E_x[I_B(x_t) \bar{G}_x(x_t)] = \sum_{k=2}^{\infty} \int_0^t E_0[I_B(x_{t-n}) \bar{G}(x_{t-u})] H_x * H^{(k-2)}(du) \\ &= \int_B p_t(x, dy) \frac{\bar{G}(y)}{\bar{G}(x)} + \int_0^t \int_B p_{t-u}(0, dy) \bar{G}(y) H_x * R(du). \end{aligned}$$

where $p_t(x, dy) = P_x(x_t \in dy)$.

To establish the Markov property, first note that

$$\begin{aligned} (4.3) \quad P_x(x_s \in A, x_s < Y_x \leq x_{s+t}) &= E_x[I_A(x_s) (G_x(x_{s+t}) - G_x(x_s))] \\ &= E_x[I_A(x_s) \frac{\bar{G}(x_s)}{\bar{G}(x)} G_{x_s}(x_{s+t})] \\ &= \int_A p_s(x, dy) \frac{\bar{G}(y)}{\bar{G}(x)} H_y(t). \end{aligned}$$

Let

$$p_0(z_t \in dy) = q_t(0, dy) = \int_0^t p_{t-u}(0, dy) \bar{G}(y) R(du)$$

and

$$p_x(z_t \in dy) = p_t(x, dy) \frac{\bar{G}(y)}{\bar{G}(x)} + \int_0^t p_{t-u}(0, dy) \bar{G}(y) H_x * R(du)$$

The Markov property for $z_t: t \geq 0$ will follow if we establish

$$P_x(z_s \in A, z_{s+t} \in B) = \int_B \int_A q_s(x, dy) q_t(y, dz).$$

Now

$$\begin{aligned} (4.4) \quad P_x(x_s^1 \in A, x_s^1 < Y_x^1, x_{s+t}^1 \in B, x_{s+t}^1 < Y_x^1) \\ = \int_A p_s(x, dy) \int_B p_t(y, dz) \frac{\bar{G}(t)}{\bar{G}(y)} \\ = \int_A p_s(x, dy) \frac{\bar{G}(y)}{\bar{G}(x)} \int_B p_t(y, dz) \frac{\bar{G}(z)}{\bar{G}(x)} \end{aligned}$$

By (4.3) we have

$$\begin{aligned} (4.5) \quad \sum_{j=2}^{\infty} P_{x,0}(x_s^1 \in A, x_s^1 < Y_x^1, \sum_{\ell=0}^{j-1} \sigma_{\ell} \leq s+t, x_{s+t-j-1}^j \in B, x_{s+t-j-1}^j < Y^j) \\ = \sum_{j=2}^{\infty} \int_A p_s(x, dy) \frac{\bar{G}(y)}{\bar{G}(x)} \int_0^t \int_B p_{t-u}(0, dz) \bar{G}(z) H_y^{*(j-2)}(du) \\ = \int_A p_s(x, dy) \frac{\bar{G}(y)}{\bar{G}(x)} \int_0^t \int_B p_{t-u}(0, dz) \bar{G}(z) H_y^* R(du) \end{aligned}$$

Also

$$\begin{aligned} (4.6) \quad \sum_{k=2}^{\infty} P_{x,0} \left(\sum_{\ell=0}^{k-1} \sigma_{\ell} \leq s, x_{s-k-1}^k \in A, x_{s-k-1}^k < Y^k, \sum_{\ell=0}^{k-1} \sigma_{\ell} \leq s+t, \right. \\ \left. x_{s+t-k-1}^k \in B, x_{s+t-k-1}^k < Y^k \right) \\ = \sum_{k=2}^{\infty} \int_0^s \int_A p_{s-u}(0, dy) \bar{G}(y) \int_B p_t(y, dz) \frac{\bar{G}(y)}{\bar{G}(z)} H_x^{*(k-2)}(du) \\ = \int_0^s \int_A p_{s-u}(0, dy) \bar{G}(y) \int_B p_t(y, dz) \frac{\bar{G}(z)}{\bar{G}(y)} H_x^* R(du). \end{aligned}$$

Again by (4.2)

$$\begin{aligned}
 (4.7) \quad & \sum_{k=2}^{\infty} \sum_{j=k+1}^{\infty} P_{x,0} \left(\sum_{\ell=0}^{k-1} \sigma_{\ell} \leq s, x_{s-k-1}^k \in A, x_{s-k-1}^k < Y_x, \right. \\
 & \left. \sum_{\ell=0}^j \sigma_{\ell} \leq s+t, x_{s+t-j-1}^j \in B, x_{s+t-j-1}^j < Y^j \right) \\
 & = \sum_{k=2}^{\infty} \sum_{j=k+1}^{\infty} \int_0^s \int_A p_{s-u}(0, dy) \bar{G}(y) \int_0^t \int_B p_{t-u}(0, dz) \bar{G}(z) H_y * H^{(j-k-1)}(du) \\
 & \quad H_x * H^{(k-2)}(du) \\
 & = \int_0^s \int_A p_{s-u}(0, dy) \bar{G}(y) \int_0^t \int_B p_{t-u}(0, dt) \bar{G}(z) H_y * R(du) H_x * R(du)
 \end{aligned}$$

Now by combining (4.4)-(4.7),

$$\begin{aligned}
 P_x(z_s \in A, z_{t+s} \in B) &= \sum_{j=1}^{\infty} \sum_{k=j}^{\infty} P_{x,0} \left(\sum_{\ell=0}^{k-1} \sigma_{\ell} \leq s, x_{s-k-1}^k \in A, \right. \\
 & \quad \left. x_{s-k-1}^k < Y^k, \sum_{\ell=0}^{k-1} \sigma_{\ell} \leq s+t, x_{t+s-j-1}^j \in B, x_{t+s-j-1}^j < Y^j \right) \\
 &= \int_A p_s(x, dy) \frac{\bar{G}(y)}{\bar{G}(x)} \int_B p_t(y, dz) \frac{\bar{G}(z)}{\bar{G}(x)} + \int_0^t \int_B p_{t-u}(0, dz) \bar{G}(z) H_y * R(du) \\
 &= \int_0^s \int_A p_{s-u}(0, dy) \bar{G}(y) \int_B p_t(y, dz) + \int_0^t \int_B p_{t-u}(0, dz) \bar{G}(z) H_y * R(du) H_x * R(du) \\
 &= \int_A \int_B q_s(x, dy) q_t(y, dz)
 \end{aligned}$$

Next let

$$S_t f(x) = E_x[f(z_t)] = \frac{1}{\bar{G}(x)} T_t(\bar{G}f)(x) + \int_0^t T_{t-s}(\bar{G}f)(0) H_x * R(ds)$$

For $f \in C_b(R^+)$, since $G(x)$ is assumed continuous and $T_t: t \geq 0$ is Feller

$$\frac{1}{G(x)} T_t(\bar{G}f)(x) \in C_b(R^+)$$

Moreover

$H_x(t) = E_x[G_x(X_t)]$ is continuous in x and therefore the family of distributors $\{H_x(t): x \in R^+\}$ is continuous in distribution in x . Since $T_{t-s}(\bar{G}f)(0)$ is continuous in s and $H_x * R(t)$ is continuous in t ,

$$\int_0^t T_{t-s}(\bar{G}f)(0) H_x * R(ds) \text{ is continuous in } x \text{ and so } S_t f \in C_b(R^+).$$

To show that $S_t: t \geq 0$ is strongly continuous we assume

$$(4.8) \quad G \in D_A \text{ and } \frac{AG(x)}{G(x)} \text{ is continuous and bounded.}$$

Note that

$$(4.9) \quad \int_0^t T_{t-s}(\bar{G})(0) H_x * R(ds) = E_x[G_x(x_t)].$$

so

$$|S_t f(x) - f(x)| \leq |T_t f(x) - f(x)| + 2\|f\| T_t G_x(x)$$

From (4.8) by Dynkins formula

$$\begin{aligned} 0 \leq T_t G_x(x) &\leq \frac{1}{G(x)} (T_t G(x) - G(x)) = \frac{1}{G(x)} \int_0^t T_s AG(x) \\ &\leq t \left\| \frac{AG}{G} \right\| \end{aligned}$$

and therefore

$$\|S_t f - f\| \leq \|T_t f - f\| + 2\|f\| \left\| \frac{AG}{G} \right\| t$$

and we have strong continuity.

To compute the infinitesimal generator, observe that

$$\begin{aligned} \tilde{A}f(x) &= \lim_{t \rightarrow 0} \frac{S_t f(x) - f(x)}{t} \\ &= \lim_{t \rightarrow 0} \frac{1}{G(x)} \frac{T_t(\bar{G}f)(x) - \bar{G}(x)f(x)}{t} + \lim_{t \rightarrow 0} \frac{1}{t} \int_0^t p_{t-u}(0, dy) \bar{G}(y)f(y) H_x * R(du) \end{aligned}$$

Pointwise then

$$\lim_{t \rightarrow 0} \frac{1}{\bar{G}(x)} \frac{T_t(\bar{G}f)(x) - \bar{G}(x)f(x)}{t} = \frac{A(Gf)(x)}{\bar{G}(x)}$$

and

$$\begin{aligned} \lim_{t \rightarrow 0} \frac{1}{t} \int_0^t p_{t-u}(0, dy) \bar{G}(y) f(y) H_x * R(du) &= \lim_{t \rightarrow 0} \frac{1}{t} \int_0^t p_{t-u}(0, dy) \bar{G}(y) (f(y) - f(0)) H_x * R(du) \\ (4.10) \quad &+ f(0) \lim_{t \rightarrow 0} \frac{1}{t} \int_0^t p_{t-u}(0, dy) \bar{G}(y) H_x * R(du). \end{aligned}$$

Clearly the first term on the right of (4.10) goes to 0 and by (4.9)

$$\begin{aligned} \lim_{t \rightarrow 0} \frac{1}{t} \int_0^t p_{t-u}(0, dy) \bar{G}(y) H_x * R(du) &= \lim_{t \rightarrow 0} \frac{1}{\bar{G}(x)} \frac{T_t G(x) - G(x)}{t} \\ &= \frac{AG(x)}{\bar{G}(x)}. \end{aligned}$$

Therefore pointwise

$$(4.11) \quad \tilde{A}f(x) = \frac{A(Gf)(x)}{\bar{G}(x)} + f(0) \frac{AG(x)}{\bar{G}(x)}$$

With respect to supremum norm on $C_b(R^+)$ we establish the following: Under the condition (4.8), $f \in D_A$ if and only if $\bar{G}f \in D_A$ and $\frac{A\bar{G}f}{\bar{G}}$ is bounded and continuous. We first note that

$$\begin{aligned} (4.12) \quad &\left| \frac{1}{t} \int_0^t p_{t-u}(0, dy) \bar{G}(y) f(y) H_x * R(du) - f(0) \frac{AG(x)}{\bar{G}(x)} \right| \\ &\leq \left| \frac{1}{t} \int_0^t T_{t-u}(\bar{G}(f - f(0)))(0) H_x * R(du) \right| + \frac{1}{t} |f(0)| \left| \int_0^t T_{t-u} \bar{G}(0) H_x * R(du) - \frac{AG(x)}{\bar{G}(x)} \right| \end{aligned}$$

From (4.8), (4.9) and Dynkin's formula, for t small enough

$$\begin{aligned} \left\| \frac{1}{t} \int_0^t T_{t-u}(\bar{G}(f - f(0)))(0) H_x * R(du) \right\| &\leq \sup_{0 \leq s \leq t} \|T_s \bar{G}(f - f(0))\| \left\| \frac{1}{t} H_x * R(t) \right\| \\ &\leq \sup_{0 \leq s \leq t} \|T_s(\bar{G}(f - f(0)))\| \left\| \frac{1}{t} E_x \frac{[G(x_t) - G(x)]}{\bar{G}(x)} \right\| \\ &\leq \sup_{0 \leq s \leq t} \|T_s(\bar{G}(f - f(0)))\| C \left\| \frac{AG}{\bar{G}} \right\| \rightarrow 0. \end{aligned}$$

Moreover by (4.8), (4.9) and Dynkin's formula again,

$$\begin{aligned} & \left| \frac{1}{t} \int_0^t T_{t-u}(\bar{G})(0) H_x^* R(du) - \frac{AG(x)}{\bar{G}(x)} \right| \\ &= \left| \frac{T_t G(x) - G(x)}{\bar{G}(x)t} - \frac{AG(x)}{\bar{G}(x)} \right| \\ &\leq \frac{1}{t} \int_0^t \|T_s \left(\frac{AG}{\bar{G}} \right) - \frac{AG}{\bar{G}}\| ds + \frac{1}{t} \left\| \int_0^t T_s \left(\frac{AG}{\bar{G}} \left(\frac{\bar{G}}{\bar{G}(x)} - 1 \right) \right) ds \right\| \end{aligned}$$

Now the first term goes to 0 and for the second,

$$\begin{aligned} (4.13) \quad \frac{1}{t} \left\| \int_0^t T_s \left(\frac{AG}{\bar{G}} \left(\frac{\bar{G}}{\bar{G}(x)} - 1 \right) \right) ds \right\| &\leq \left\| \frac{AG}{\bar{G}} \right\| \frac{1}{t} \int_0^t \left\| \frac{T_s G - G}{\bar{G}} \right\| ds \\ &\leq \left\| \frac{AG}{\bar{G}} \right\|^2 \frac{1}{t} \int_0^t s ds \end{aligned}$$

and so we have uniform convergence for the left hand side of (4.12).

Next if $f \in D_{\bar{A}}$, then by (4.11) and the fact that we have uniform convergence in (4.12)

$$\begin{aligned} & \left\| \frac{1}{t} (T_t(\bar{G}f) - \bar{G}f) - \frac{\bar{A}Gf}{\bar{G}} \right\| \leq \left\| \frac{T_t(\bar{G}f) - \bar{G}f}{t\bar{G}} - \frac{\bar{A}Gf}{\bar{G}} \right\| \\ &\leq \left\| \frac{S_t f - f}{t} - \bar{A}f \right\| + \left\| \int_0^t T_{t-u}(\bar{G}f)(0) H_x^* R(du) - f(0) \frac{AG(x)}{\bar{G}(x)} \right\| \end{aligned}$$

and hence

$$\lim_{t \rightarrow 0} \left\| \frac{1}{t} (T_t(\bar{G}f) - \bar{G}f) - \frac{\bar{A}Gf}{\bar{G}} \right\| = 0$$

That $\frac{\bar{A}Gf}{\bar{G}}$ is bounded follows from (4.11).

Conversely if $\bar{G}f \in D_{\bar{A}}$ and $\frac{\bar{A}Gf}{\bar{G}}$ is bounded to show

$$\begin{aligned} & \left\| \frac{S_t f - f}{t} - \bar{A}f \right\| \rightarrow 0 \quad \text{we must show} \\ & \left\| \frac{T_t(\bar{G}f) - \bar{G}f}{\bar{G}} - \frac{A(Gf)}{\bar{G}} \right\| \rightarrow 0. \end{aligned}$$

This is so since

$$\begin{aligned} \left\| \frac{T_t(\bar{G}f) - \bar{G}f}{\bar{G}} - \frac{A(\bar{G}f)}{\bar{G}} \right\| &\leq \frac{1}{t} \int_0^t \left\| T_s \left(\frac{A(\bar{G}f)}{\bar{G}} \right) - \frac{A(\bar{G}f)}{\bar{G}} \right\| ds \\ &+ \left\| \frac{A(\bar{G}f)}{\bar{G}} \right\| \frac{1}{t} \int_0^t \left\| \frac{T_s(G) - G}{\bar{G}} \right\| ds \rightarrow 0. \end{aligned}$$

The latter follows as in (4.13).

To summarize, we have established:

Theorem 4.1. Assume the conditions (2.1) and (4.8). Let

$$z_t = x_{t - \sum_{\ell=0}^{k-1} \sigma_\ell} \quad \text{on the set} \quad \sum_{\ell=0}^{k-1} \sigma_\ell \leq t < \sum_{\ell=0}^k \sigma_\ell.$$

Then $z_t: t \geq 0$ is a strong Markov process and has transition probabilities given by

$$q_t(x, dy) = p_t(x, dy) \frac{\bar{G}(y)}{\bar{G}(x)} + \int_0^t p_{t-u}(0, dy) \bar{G}(y) H_x^* R(du)$$

Moreover the associated semigroup $S_t: t \geq 0$ acting on $C_b(R^+)$ is strongly continuous and Feller

Further letting $\tilde{A}f$ denote the infinitesimal generator then $f \in D_{\tilde{A}}$ if and only if $\bar{G}f \in D_A$ and $\frac{\tilde{A}(\bar{G}f)}{\bar{G}}$ is bounded. Lastly

$$\tilde{A}f(x) = \frac{A(\bar{G}f)(x)}{\bar{G}(x)} + f(0) \frac{AG(x)}{\bar{G}(x)}.$$

Theorem 4.2. Suppose (2.1), (4.13) and

$$(4.14) \quad E_x[\sigma_x] < \infty \quad \text{for all } x.$$

Then $z_t: t \geq 0$ has a unique invariant probability measure Π given by

$$\Pi(A) = \frac{E_0 \left[\int_0^\sigma I_A(x_t) dt \right]}{E_0[\sigma]} = \frac{E_0 \left[\int_0^\infty I_A(x_t) \bar{G}(x_t) dt \right]}{E_0 \left[\int_0^\infty \bar{G}(x_t) dt \right]}.$$

Proof. First note that by Lemma 3.3, the two representations of Π are equivalent. Recall the notation $\sigma_1 = \sigma_x = \inf\{t: x_t^1 \geq Y_x^1\}$ and $\sigma_k = \inf\{t: x_t^k \geq Y_x^k\}$. Since $\sigma_k: k \geq 1$ are independent as well as $\int_0^{\sigma_k} f(x_t^k) dt: k \geq 1$ and for $k \geq 2$, are respectively identically distributed with means

$$E[\sigma_k] = E_0 \left[\int_0^\infty \bar{G}(x_t) dt \right] \quad k \geq 2$$

and

$$E \left[\int_0^{\sigma_k} f(x_t^k) dt \right] = E_0 \left[\int_0^\infty f(x_t) \bar{G}(x_t) dt \right] \quad k \geq 2$$

we have by the Strong Law of Large Number for $f \in C_b(\mathbb{R}^+)$

$$\begin{aligned} \lim_{t \rightarrow \infty} \frac{1}{t} \int_0^t f(z_s) ds &= \lim_{n \rightarrow \infty} \frac{\frac{1}{n} \sum_{k=1}^n \int_0^{\sigma_k} f(x_s^k) ds}{\frac{1}{n} \sum_{k=1}^n \sigma_k} \\ &= \frac{E_0 \left[\int_0^\infty f(x_t) \bar{G}(x_t) dt \right]}{E_0 \left[\int_0^\infty \bar{G}(x_t) dt \right]} \quad \text{a.s. } P_z \text{ for all } z. \end{aligned}$$

Hence Π is an invariance measure.

If $\tilde{\Pi}$ is any other invariant measure, then for $f \in C_b(\mathbb{R}^+)$,

$$\int f(z) \tilde{\Pi}(dz) = \int E_z[f(z_s)] \tilde{\Pi}(dz)$$

Therefore by Lebesgue Dominated Convergence

$$\begin{aligned} \int f(z) \tilde{\Pi}(dz) &= \lim_{t \rightarrow \infty} \frac{1}{t} \int_0^t \int E_z[f(z_s)] \tilde{\Pi}(dz) ds \\ &= \int E_z \left[\lim_{t \rightarrow \infty} \frac{1}{t} \int_0^t f(z_s) ds \right] \tilde{\Pi}(dz) = \int f(z) \Pi(dz) \end{aligned}$$

Hence $\tilde{\Pi} = \Pi$.

Remark 4.1. In the work of Robin [8] it is required that

$$P_x(z_t \in \Gamma) - \Pi(\Gamma) \leq B e^{-\gamma t} \quad \gamma > 0$$

and as a consequence

$$S_t g(x) - \Pi(g) \leq B_1 e^{-\gamma_1 t} \quad \gamma > 0$$

where

$$\Pi(g) = \int g(x) \Pi(dx).$$

For our case we have only that

$$\lim_{t \rightarrow 0} \frac{1}{t} \int_0^t S_u g(x) du = \Pi(g).$$

Since $z_t: t \geq 0$ is periodic, we will not even have pointwise convergence.

Let $f \in C_b(R^+)$ and $u_\alpha(x)$ be the unique solution of $\tilde{A}u_\alpha - \alpha u_\alpha = -f(x)$.

It is standard that

$$(4.15) \quad u_\alpha(x) = \int_0^\infty e^{-\alpha t} q_t(x, dy) f(y).$$

Define

$$\bar{f} = \Pi(f) = \frac{E_c \left[\int_0^\infty \bar{G}(x_s) f(x_s) ds \right]}{E_o \left[\int_0^\infty \bar{G}(x_s) ds \right]}.$$

Theorem 4.3. Under the assumption (2.1), (4.8) and the assumption

$$(4.16) \quad E_x [\sigma_x] = \frac{E_x \left[\int_0^\infty \bar{G}(x_s) ds \right]}{\bar{G}(x)} \leq C \text{ independent of } x,$$

$$(4.17) \quad P_x(X_t < \infty) = 1 \text{ for all } t \text{ and } x, \text{ and}$$

$$\lim_{t \rightarrow \infty} x_t = \infty \quad \text{a.s. } P_x \text{ for all } x.$$

$$(i) \quad \lim_{\alpha \rightarrow 0} \alpha u_{\alpha}(x) = \bar{f}$$

$$(ii) \quad \text{Let } v_{\alpha}(x) = u_{\alpha}(x) - u_{\alpha}(0). \text{ Then}$$

$$\lim_{\alpha \rightarrow 0} v_{\alpha}(x) = \bar{v}(x) \text{ uniformly on compact sets where}$$

$$\bar{v}(x) = \frac{E_x \left[\int_0^{\infty} (f(x_s) - \bar{f}) \bar{G}(x_s) ds \right]}{\bar{G}(x)}$$

and \bar{v} is the unique solution of

$$-\bar{A}v = f - \bar{f} \text{ with } \bar{v}(0) = 0.$$

Proof. From (4.2) and (4.15)

$$u_{\alpha}(x) = \frac{1}{\bar{G}(x)} E_x \left[\int_0^{\infty} e^{-\alpha t} f(x_t) \bar{G}(x_t) dt \right]$$

$$+ \int_0^{\infty} e^{-\alpha t} \int_0^t E_0 \left[f(x_{t-s}) \bar{G}(x_{t-s}) \right] H_x * R(ds) dt.$$

By (4.16)

$$\left| \frac{1}{\bar{G}(x)} E_x \left[\int_0^{\infty} e^{-\alpha t} f(x_t) \bar{G}(x_t) dt \right] \right| \leq f \frac{E_x \left[\int_0^{\infty} \bar{G}(x_t) dt \right]}{\bar{G}(x)}$$

and so

$$\lim_{\alpha \rightarrow 0} \alpha \left| \frac{1}{\bar{G}(x)} E_x \left[\int_0^{\infty} e^{-\alpha t} f(x_t) \bar{G}(x_t) dt \right] \right| \rightarrow 0$$

Next by the convolution property of Laplace transforms,

$$\begin{aligned} & \int_0^\infty e^{-\alpha t} \int_0^t E_0 \left[f(x_{t-s}) \bar{G}(x_{t-s}) \right] H_x * R(ds) dt \\ &= \int_0^\infty e^{-\alpha t} E_0 [\bar{G}(x_t) f(x_t)] dt \int_0^\infty e^{-\alpha t} H_x(dt) \int_0^\infty e^{-\alpha t} R(dt) \end{aligned}$$

Let

$$\hat{H}_x(\alpha) = \int_0^\infty e^{-\alpha t} H_x(dt) \quad \text{and note that} \quad \hat{H}_x(\alpha) \rightarrow 1 \quad \text{as} \quad \alpha \rightarrow 0.$$

Since

$$\int_0^\infty e^{-\alpha t} R(dt) = \frac{1}{1 - \hat{H}(\alpha)} \quad \text{where} \quad \hat{H}(\alpha) = \int_0^\infty e^{-\alpha t} H(dt)$$

we have

$$\lim_{\alpha \rightarrow 0} \frac{1 - \hat{H}(\alpha)}{\alpha} = E_0[\sigma] = E_0 \left[\int_0^\infty \bar{G}(x_t) dt \right] \quad \text{showing (i).}$$

For (ii), note that

$$u_\alpha(0) = E_0 \left[\int_0^\infty e^{-\alpha t} \bar{G}(x_t) f(x_t) dt \right] \frac{1}{1 - \hat{H}(\alpha)}.$$

Therefore

$$\begin{aligned} (4.18) \quad u_\alpha(x) - u_\alpha(0) &= \frac{1}{\bar{G}(x)} E_x \left[\int_0^\infty e^{-\alpha t} \bar{G}(x_t) f(x_t) dt \right] \\ &\quad + E_0 \left[\int_0^\infty e^{-\alpha t} \bar{G}(x_t) f(x_t) dt \right] \frac{\hat{H}_x(\alpha) - 1}{1 - \hat{H}(\alpha)}. \end{aligned}$$

It is enough to establish the uniform convergence on compact set for $f \geq 0$, the general case easily follows.

Now

$$\frac{1}{\bar{G}(x)} E_x \left[\int_0^\infty e^{-\alpha t} \bar{G}(x_t) f(x_t) dt \right] + \frac{1}{\bar{G}(x)} E_x \left[\int_0^\infty \bar{G}(x_t) f(x_t) dt \right]$$

which is bounded by (4.16). Next

$$\lim_{\alpha \rightarrow 0} E_0 \left[\int_0^{\infty} e^{-\alpha t} \bar{G}(x_t) f(x_t) dt \right] \frac{\alpha}{1 - \hat{H}(\alpha)} = \bar{f}.$$

For the other part of the second term in (4.18),

$$\begin{aligned} \frac{\hat{H}_x(\alpha) - 1}{\alpha} &= \frac{E_x \left[\int_0^{\infty} \alpha e^{-\alpha t} (G(x_t) - G(x)) dt - \bar{G}(x) \right]}{\alpha \bar{G}(x)} \\ &= - \frac{E_x \left[\int_0^{\infty} e^{-\alpha t} \bar{G}(x_t) dt \right]}{\bar{G}(x)} + - \frac{E_x \left[\int_0^{\infty} \bar{G}(x_t) dt \right]}{\bar{G}(x)}. \end{aligned}$$

Therefore, uniformly on compact sets

$$\lim_{\alpha \rightarrow 0} u_{\alpha}(x) - u_{\alpha}(0) = \frac{E_x \left[\int_0^{\infty} \bar{G}(x_t) (f(x_t) - \bar{f}) dt \right]}{\bar{G}(x)}.$$

To show that

$$-\bar{A}\bar{v}(x) = f(x) - \bar{f}$$

first note that $\bar{v}(0) = 0$ and so by Theorem 4.1 it is enough to show

$$-A(\bar{G} \bar{v})(x) = \bar{G}(x)(f(x) - \bar{f}).$$

Note that by the Markov property

$$\begin{aligned} \bar{G}(x)\bar{v}(x) &= E_x \left[\int_0^{\infty} \bar{G}(x_s) (f(x_s) - \bar{f}) ds \right] \\ &= E_x \left[\int_0^t \bar{G}(x_s) (f(x_s) - \bar{f}) ds + \int_0^{\infty} \bar{G}(x_{t+s}) (f(x_{t+s}) - \bar{f}) ds \right] \\ &= E_x \left[\int_0^t \bar{G}(x_s) (f(x_s) - \bar{f}) ds + \bar{G}(x_t) \bar{v}(x_t) \right] \end{aligned}$$

And so

$$\left\| \frac{T_t \bar{G} \bar{v} - \bar{G} \bar{v}}{t} + \bar{G}(f - \bar{f}) \right\| \leq \frac{1}{t} \int_0^t \|T_s \bar{G}(f - \bar{f}) - \bar{G}(f - \bar{f})\| ds \rightarrow 0.$$

To show uniqueness, suppose \bar{v}_1 and \bar{v}_2 are two solutions of $-\tilde{A}v = f - \bar{f}$, $v(0) = 0$. Let $w = \bar{v}_1 - \bar{v}_2$. Then $\tilde{A}w = 0$ and since $w(0) = 0$ we must have $\bar{A}Gw = 0$. By Dynkin's formula

$$\bar{G}(x)w(x) = E_x[\bar{G}(x_t)w(x_t)] \text{ for all } t$$

and by (4.18) it follows that

$$\lim_{t \rightarrow \infty} E_x[\bar{G}(x_t)w(x_t)] = 0 \text{ i.e. } w(x) = 0 \text{ for all } x.$$

It is well known that a necessary condition that $\tilde{A}v = -f$ have a solution is that $\Pi(f) = 0$. See Robin [8].

Theorem 4.4. Under the assumption (2.1), (4.8), (4.16) and (4.17) a necessary and sufficient condition that

$$(4.19) \quad \tilde{A}v = -f$$

have a solution is that $\Pi(f) = 0$.

Moreover if $\Pi(f) = 0$, then any two solution of (4.19) differ by a constant.

Proof. We need only prove the sufficiency. Suppose $\tilde{A}v = -f$. Then by Theorem 4.1

$$Av = -\bar{G}f - v(0)AG.$$

Therefore by Dynkin's formula

$$v(0) = E_0 \left[\bar{G}(x_t)v(x_t) + \int_0^t \bar{G}(x_s)f(x_s)ds + v(0)G(x_t) \right]$$

Letting $t \rightarrow \infty$ (4.17)

$$v(0) = E_0 \left[\int_0^\infty \bar{G}(x_s) f(x_s) ds \right] + v(0),$$

that is $\Pi(f) = 0$.

Next suppose v_1 and v_2 are two solutions of

$$\tilde{A}v = -f.$$

Then by Theorem 4.1

$$A \bar{G}(v_1 - v_2) = -(v_1 - v_2)[0] AG$$

and so by Dynkin's formula again

$$\begin{aligned} \bar{G}(x)(v_1(x) - v_2(x)) &= E_x [\bar{G}(x_t)(v_1(x_t) - v_2(x_t))] \\ &\quad + (v_1 - v_2)(0) E_x \left[\int_0^t AG(x_s) ds \right] \end{aligned}$$

Thus by (4.17)

$$\bar{G}(x)(v_1(x) - v_2(x)) = \lim_{t \rightarrow \infty} E_x [G(x_t) - G(x)](v_1 - v_2)(0)$$

or

$$v_1(x) - v_2(x) = v_1(0) - v_2(0).$$

§5. Asymptotics of a Stopping Problem.

Assume that

$$(5.1) \quad f, \psi \in C_b(R^+) \quad \text{with} \quad f, \psi \geq 0$$

and consider the stopping problem

$$(5.2) \quad u(x) = \inf_{\tau} J_x(\tau)$$

where

$$J_x(\tau) = E_x \left[\int_0^{\tau} f(x_s) ds + \psi(x_{\tau}) I_{(\tau < \infty)} \right]$$

Define for $\alpha > 0$

$$J_x^{\alpha}(\tau) = E_x \left[\int_0^{\tau} f(x_s) ds + e^{-\alpha\tau} \psi(x_{\tau}) \right]$$

and

$$(5.3) \quad u_{\alpha}(x) = \inf_{\tau} J_x^{\alpha}(\tau).$$

See Bensoussan [3] or Robin [9] for the following:

Theorem 5.1. Under the assumption (2.1), (2.2) and (5.1)

(i) u_{α} is the maximal element of the set of functions $h \in C_b(R^+)$

$$h \leq e^{-\alpha t} T_t h + \int_0^t e^{-\alpha s} T_s f ds \quad h \leq \psi$$

(ii) $\hat{\tau}_{\alpha} = \inf\{t: u_{\alpha}(x_t) = \psi(x_t)\}$ is optimal, is

$$u_{\alpha}(x) = J_x^{\alpha}(\hat{\tau}_{\alpha}).$$

(iii) For $\epsilon > 0$, define

$$J_x^{\alpha, \epsilon}(V) = E_x \left[\int_0^{\infty} e^{-\alpha t} \exp - \frac{1}{\epsilon} \int_0^t V_s ds (f(x_t) + \frac{1}{\epsilon} V_t \psi(x_t)) dt \right]$$

where $0 \leq V_t \leq 1$ and $V_t \in F_t: t \geq 0$. Let

$$w_{\alpha, \epsilon}(x) = \inf_V J_x^{\alpha, \epsilon}(V)$$

Then $w_{\alpha, \varepsilon}$ is the unique solution in $C_b(R^+)$ of

$$w_{\alpha, \varepsilon} = \int_0^\infty e^{-\alpha t} T_t \left[f - \frac{1}{\varepsilon} (w_{\alpha, \varepsilon} - \psi)^+ \right] dt$$

and moreover if

$$\hat{V}_t = \begin{cases} 1 & w_{\alpha, \varepsilon}(x_t) \geq \psi(x_t) \\ 0 & w_{\alpha, \varepsilon}(x_t) < \psi(x_t) \end{cases}$$

then

$$w_{\alpha, \varepsilon}(x) = J_x^{\alpha, \varepsilon}(\hat{V}).$$

(iv) $w_{\alpha, \varepsilon} \downarrow v_\alpha$ as $\varepsilon \rightarrow 0$ uniformly on compact sets.

Suppose we have the additional condition

$$(5.4) \quad f(x) \geq \beta > 0.$$

Remark 5.1. For the consideration of (5.1), under (5.3), we can restrict our attention to stopping times τ so that

$$E_x(\tau) \leq \frac{\|\psi\|}{\beta}$$

since by taking $\tau \equiv 0$, we obtain $u(x) \leq \|\psi\|$ and for any stopping time τ , $\beta E_x[\tau] \leq J_x^\alpha(\tau)$.

For the following see Robin [8], Theorem 3.1 and Remark 3.3:

Theorem 5.2. Under (2.1), (2.2), (5.1) and (5.4)

(i) u is the maximal element of the set of functions

$$h \in C_b(R^+) \quad h \leq T_t h + \int_0^t T_s f ds \quad h \leq \psi$$

(ii) $\hat{\tau} = \inf\{t: u(x_t) = \psi(x_t)\}$ is optimal, i.e.

$$u(x) = J_x(\hat{\tau})$$

(iii) $u_\alpha \uparrow u$ uniformly on compact sets as $\alpha \rightarrow 0$

(iv) Let $\varepsilon > 0$ and

$$w_\varepsilon(x) = \inf_V J_x^\varepsilon(V)$$

where

$$J_x^\varepsilon(V) = E_x \left[\int_0^\infty \exp \left(-\frac{1}{\varepsilon} \int_0^t V_s ds \right) (f(x_t) + \frac{1}{\varepsilon} V_t \psi(x_t)) dt \right]$$

and $0 \leq V_s \leq 1$, $V_t \in F_t : t \geq 0$. Then w_ε is the unique solution in $C_b(R^+)$ of

$$w_\varepsilon = T_t w_\varepsilon + \int_0^t T_s (f - \frac{1}{\varepsilon} (w_\varepsilon - \psi)^+) ds$$

and if

$$\hat{V}_t = \begin{cases} 1 & w_\varepsilon(x_t) \geq \psi(x_t) \\ 0 & w_\varepsilon(x_t) < \psi(x_t) \end{cases}$$

then

$$w_\varepsilon(x) = J_x^\varepsilon(\hat{V}).$$

(v) $w_{\alpha, \varepsilon} \uparrow w_\varepsilon$ uniformly on compact sets.

(vi) $w_\varepsilon \downarrow u$ uniformly on compact sets.

Define for $\alpha > 0$

$$J_x^\alpha(\tau) = E_x \left[\int_0^\tau \bar{G}(x_t) f(x_t) dt + e^{-\alpha \tau} \bar{G}(x_\tau) \psi(x_\tau) \right]$$

where f and ψ satisfy (5.1) and (5.4) and $G(x)$ is a continuous distribution function as before. Let

$$J_x(\tau) = E_x \left[\int_0^\tau \bar{G}(x_t) f(x_t) dt + \bar{G}(x_\tau) \psi(x_\tau) \right].$$

Under the assumption (4.17), it is consistent to define

$$\bar{G}(x_\tau) \psi(x_\tau) = 0 \quad \text{on the set } (\tau = \infty).$$

Hence under (4.16), $J_x(\tau) < \infty$ for any stopping time τ and moreover by quasi-left continuity, see (2.2)

$$(5.5) \quad \lim_{t \rightarrow \infty} J_x(\tau \wedge t) = J_x(\tau) \quad \text{and} \quad \lim_{t \rightarrow \infty} J_x^\alpha(\tau \wedge t) = J_x^\alpha(\tau).$$

Let

$$u(x) = \inf_{\tau} J_x(\tau) \quad \text{and} \quad u_{\alpha}(x) = \inf J_x^{\alpha}(\tau).$$

We wish to establish the following generalization of Theorem 5.2:

Theorem 5.3. Under (2.1), (2.2), (4.16), (4.17), (5.1), and (5.4)

(i) u is the maximal element of the set of functions

$$h \in C_b(\mathbb{R}^+), \quad h \leq T_t h + \int_0^t T_s(\bar{G}f)ds \quad h \leq \bar{G}\psi.$$

(ii) $\hat{\tau} = \inf\{t: u(x_t) = \bar{G}(x_t)\psi(x_t)\}$ is optimal, i.e.

$$u(x) = J_x(\hat{\tau})$$

(iii) $u_{\alpha} \uparrow u$ uniformly on compact sets.

The proof will follow after a series of lemmas. Let

$$J_x^n(\tau) = E_x \left[\int_0^{\tau} (\bar{G}(x_t) V \frac{1}{n} f(x_t) dt + \bar{G}(x_{\tau}) \psi(x_{\tau})) \right]$$

and

$$u_n(x) = \inf_{\tau} J_x^n(\tau).$$

Lemma 5.4. $u_n \in C_b(\mathbb{R}^+)$ and $u_n(x) \uparrow u(x)$.

Proof. Note that by (5.4) and Remark 5.1

$$u_n(x) = \inf\{J_x(\tau): E_x[\tau] < \infty\}.$$

Since for any τ with $E_x[\tau] < \infty$

$$J_x^n(\tau) \uparrow J_x(\tau)$$

it follows that

$$u_n(x) \uparrow \tilde{u}(x) \quad \tilde{u}(x) \geq u(x).$$

Now Theorem 5.4 applies to $u_n(x)$ and so for any stopping time τ with $E_x[\tau] < \infty$

$$u_n(x) \leq J_x^n(\tau)$$

Letting $n \rightarrow \infty$

$$\tilde{u}(x) \leq J_x(\tau) \quad \text{and so} \quad \tilde{u}(x) \leq u(x),$$

$$\text{i.e.} \quad \tilde{u}(x) = u(x).$$

Let

$$J_x^{\varepsilon, n}(V) = E_x \left[\int_0^\infty e^{-1/\varepsilon \int_0^t V_s ds} \left(\bar{G}(x_t) V \frac{1}{n} \right) f(x_t) + \frac{1}{\varepsilon} V_t \bar{G}(x_t) \psi(x_t) dt \right]$$

$$J_x^{\varepsilon, \alpha}(V) = E_x \left[\int_0^\infty e^{-\alpha t} e^{-1/\varepsilon \int_0^t V_s ds} \bar{G}(x_t) f(x_t) + \frac{1}{\varepsilon} V_t \bar{G}(x_t) \psi(x_t) dt \right]$$

and

$$J_x^\varepsilon(V) = E_x \left[\int_0^\infty e^{-1/\varepsilon \int_0^t V_s ds} \bar{G}(x_t) f(x_t) + \frac{1}{\varepsilon} V_t \bar{G}(x_t) \psi(x_t) dt \right]$$

where

$V_t \in F_t: t \geq 0$ with $0 \leq V_t \leq 1$. Define

$$u_{\varepsilon, n}(x) = \inf_V J_x^{\varepsilon, n}(V), \quad u_{\varepsilon, \alpha}(x) = \inf_V J_x^{\varepsilon, \alpha}(V), \quad \text{and} \quad u_\varepsilon(x) = \inf_V J_x^\varepsilon(V).$$

Lemma 5.5.

$$(i) \quad u_{\varepsilon, n} \in C_b^+(R^+) \quad \text{and} \quad u_{\varepsilon, n}(x) \uparrow u_\varepsilon(x)$$

$$(ii) \quad u_{\varepsilon, \alpha}(x) \in C_b^+(R^+) \quad \text{and} \quad u_{\varepsilon, \alpha}(x) \uparrow u_\varepsilon(x)$$

$$(iii) \quad u_\varepsilon(x) \text{ is the unique non-negative solution in } C_b^+(R^+) \text{ of}$$

$$u_\varepsilon = T_t u_\varepsilon + \int_0^t T_s (\bar{G}f - 1/\varepsilon (u_\varepsilon - \bar{G}\psi)^+) ds$$

and if

$$\hat{V}_t = \begin{cases} 1 & u_\varepsilon(x_t) \geq \bar{G}(x_t) \psi(x_t) \\ 0 & u_\varepsilon(x_t) < \bar{G}(x_t) \psi(x_t) \end{cases}$$

$$\text{then} \quad u_\varepsilon(x) = J_x^\varepsilon(\hat{V}).$$

Proof. We first claim

$$u_\varepsilon(x) = \inf \{ J_x^\varepsilon(V) : \int_0^\infty V_t dt = \infty \text{ a.s. } P_x \}$$

Suppose $P_x \left(\int_0^\infty V_t dt < \infty \right) > 0$ and let $\delta > 0$. It is enough to show that there is

a $\tilde{V}_t : t \geq 0$ so that

$$\int_0^\infty \tilde{V}_t dt = \infty \text{ a.s. } P_x \text{ and}$$

$$J_x^\varepsilon(\tilde{V}) \leq J_x^\varepsilon(V) + \delta$$

By (4.16)

$$J_x^\varepsilon(V) \leq E_x \left[\int_0^\infty \bar{G}(x_t) dt \right] (\|f\| + 1/\varepsilon \|\psi\|) < \infty$$

there is a T_0 so if $T \geq T_0$.

$$(5.6) \quad E_x \left[\int_T^\infty e^{-1/\varepsilon \int_0^t V_s ds} (\bar{G}(x_t) f(x_t) + 1/\varepsilon V_t \bar{G}(x_t) \psi(x_t)) dt \right] < \delta/2$$

Not that (4.17) implies that

$$\lim_{t \rightarrow \infty} E_x [\bar{G}(x_t)] = 0$$

and so there is a T_1 so that if $T \geq T_1$

$$E_x [\bar{G}(x_T)] < \delta/2 \|\psi\|.$$

For $T \geq T_0 \vee T_1$, define

$$V_t = \begin{cases} V_t & t \leq T \\ 1 & t > T \end{cases}$$

Now

$$\begin{aligned} J_x^\varepsilon(\tilde{V}) &= E_x \left[\int_0^T e^{-1/\varepsilon \int_0^t \tilde{V}_s ds} \bar{G}(x_t) f(x_t) + 1/\varepsilon \tilde{V}_t \bar{G}(x_t) \psi(x_t) dt \right] \\ &\quad + E_x \left[\int_T^\infty e^{-1/\varepsilon \int_0^t \tilde{V}_s ds} \bar{G}(x_t) f(x_t) + 1/\varepsilon \tilde{V}_t \bar{G}(x_t) \psi(x_t) dt \right] \end{aligned}$$

Moreover

$$E_x \left[\int_T^\infty e^{-1/\varepsilon} \int_0^t \tilde{V}_s ds \bar{G}(x_t) f(x_t) dt \right] \leq E_x \left[\int_T^\infty e^{-1/\varepsilon} \int_0^t V_s ds \bar{G}(x_t) f(x_t) dt \right]$$

and

$$E_x \left[\int_T^\infty e^{-1/\varepsilon} \int_0^t \tilde{V}_s ds \frac{1}{\varepsilon} \tilde{V}_t \bar{G}(x_t) \psi(x_t) dt \right] \leq \|\psi\| E_x \left[e^{-1/\varepsilon} \int_0^T V_s ds \bar{G}(x_t) \right] < \delta/2$$

Also by (5.6)

$$E_x \left[\int_T^\infty e^{-1/\varepsilon} \int_0^t V_s ds \frac{1}{\varepsilon} V_t \bar{G}(x_t) \psi(x_t) dt \right] < \delta/2$$

and thus

$$J_x^\varepsilon(\tilde{V}) \leq J_x^\varepsilon(V) + \delta.$$

To show (i), note that if $P_x \left(\int_0^\infty V_t dt < \infty \right) > 0$ then

(5.1) and (5.4) imply

$$J_x^{\varepsilon,n}(V) \geq \beta/n E_x \left[\int_0^\infty e^{-1/\varepsilon} \int_0^t V_s ds dt \right] = \infty$$

Also for $V_t \equiv 1$

$$J_x^{\varepsilon,n}(V) \leq \varepsilon/n \|f\| + \|\psi\| < \infty$$

and hence

$$u_{\varepsilon,n}(x) = \inf \left\{ J_x^{\varepsilon,n}(V) : \int_0^\infty V_t dt = \infty \text{ a.s. } P_x \right\}$$

Now note that if $P_x \left(\int_0^\infty V_t dt = \infty \right) = 1$ then

$$(5.7) \quad J_x^{\varepsilon,n}(V) + J_x(V)$$

and so $u_{\varepsilon,n}(x) + \tilde{u}_\varepsilon(x)$ where $\tilde{u}_\varepsilon(x) \geq u_\varepsilon(x)$.

Since

$$u_{\varepsilon,n}(x) \leq J_x^{\varepsilon,n}(V)$$

and for $V_t: t \geq 0$ with $P_x \left(\int_0^\infty V_t dt = \infty \right) = 1$ by (5.7)

we have $\tilde{u}_\varepsilon(x) = u_\varepsilon(x)$. That $u_{\varepsilon,n} \in C_b(R^+)$

follows since Theorem 5.2 (iv) applies.

For (ii) first note that $J_x^\varepsilon(V) < \infty$ for all $V_t: t \geq 0$

and

$$J_x^{\varepsilon,\alpha}(V) \uparrow J_x^\varepsilon(V) \quad \text{as } \alpha \rightarrow 0. \quad \text{Hence}$$

$$u_{\varepsilon,\alpha}(x) \uparrow \tilde{u}_\varepsilon(x) \quad \tilde{u}_\varepsilon(x) \leq u_\varepsilon(x).$$

By taking $V_t \equiv 1$ we see that

$$\begin{aligned} \tilde{u}_\varepsilon(x) \leq u_\varepsilon(x) &\leq \bar{G}(x) E_x \left[\int_0^\infty e^{-1/\varepsilon t} (f(x_t) + 1/\varepsilon \psi(x_t)) dt \right] \\ &\leq \bar{G}(x) (\varepsilon \|f\| + \|\psi\|) \end{aligned}$$

and so

$$(5.8) \quad \lim_{x \rightarrow \infty} \tilde{u}^\varepsilon(x) = 0.$$

By Theorem 5.1

$$u_{\varepsilon,\alpha}(x) = e^{-\alpha t} T_t u_{\varepsilon,\alpha}(x) + \int_0^t e^{-\alpha s} T_s (\bar{G}f - 1/\varepsilon (u_{\varepsilon,\alpha} - \bar{G}\psi)^+)(x) ds$$

and so

$$\tilde{u}_\varepsilon(x) = T_t \tilde{u}_\varepsilon(x) + \int_0^t T_s (\bar{G}f - 1/\varepsilon (\tilde{u}_{\varepsilon,\alpha} - \bar{G}\psi)^+)(x) ds$$

For any $V_t: t \geq 0$ integrating by parts yields

$$\begin{aligned} e^{-1/\varepsilon} \int_0^t V_s ds \tilde{u}_\varepsilon(x_t) &+ \int_0^t e^{-1/\varepsilon} \int_0^s V_u du \bar{G}(x_s) f(x_s) + 1/\varepsilon V_s \tilde{u}_\varepsilon(x_s) \\ &- 1/\varepsilon (\tilde{u}_\varepsilon - \bar{G}\psi)^+(x_s) ds \end{aligned}$$

is a martingale. Taking

$$\hat{V}_t = \begin{cases} 0 & \tilde{u}_\varepsilon(x_t) < \bar{G}(x_t)\psi(x_t) \\ 1 & \tilde{u}_\varepsilon(x_t) \geq \bar{G}(x_t)\psi(x_t) \end{cases}$$

yields

$$\tilde{u}_\varepsilon(x) = E_x \left[e^{-1/\varepsilon \int_0^t \hat{V}_s ds} \tilde{u}_\varepsilon(x_t) + \int_0^t e^{-1/\varepsilon \int_0^s \hat{V}_u du} \bar{G}(x_s) f(x_s) + 1/\varepsilon \hat{V}_s \bar{G}(x_s) \psi(x_s) ds \right]$$

and by (4.17) and (5.8) it follows by letting $t \rightarrow \infty$ that

$$\tilde{u}_\varepsilon(x) = J_x(\hat{V}) \quad \text{and so} \quad \tilde{u}_\varepsilon(x) = u(x).$$

What is left to prove in (iii) is to show the uniqueness. If $w_\varepsilon \in C_b(R^+)$ is any non-negative solution of

$$w_\varepsilon = T_t w_\varepsilon + \int_0^t T_s (\bar{G}f - 1/\varepsilon (w_\varepsilon - \bar{G}\psi)^+) ds$$

then as above for any $V_t: t \geq 0$

$$(5.9) \quad w_\varepsilon(x) = E_x \left[e^{-1/\varepsilon \int_0^t V_s ds} w_\varepsilon(x_t) + \int_0^t e^{-1/\varepsilon \int_0^s V_u du} \bar{G}(x_s) f(x_s) + 1/\varepsilon V_s w_\varepsilon(x_s) - 1/\varepsilon (w_\varepsilon - \bar{G}\psi)^+(x_s) ds \right]$$

Taking $V_t \equiv 1$ yields

$$0 \leq w_\varepsilon(x) \leq E_x \left[e^{-t/\varepsilon} w_\varepsilon(x_t) + \int_0^t e^{-s/\varepsilon} \bar{G}(x_s) f(x_s) + 1/\varepsilon \bar{G}(x_s) \psi(x_s) ds \right]$$

and letting $t \rightarrow \infty$ we obtain

$$0 \leq w_\varepsilon(x) \leq E_x \left[\int_0^\infty e^{-t/\varepsilon} \bar{G}(x_t) f(x_t) + 1/\varepsilon \bar{G}(x_t) \psi(x_t) dt \right] \\ \leq \bar{G}(x) (E_x[f] + \psi).$$

Thus $\lim_{x \rightarrow \infty} w_\epsilon(x) = 0$.

From (5.9) it follows for any V

$$w_\epsilon(x) \leq E_x \left[e^{-1/\epsilon} \int_0^t V_s ds w_\epsilon(x_t) + \int_0^t e^{-1/\epsilon} \int_0^s V_u du \bar{G}(x_s) f(x_s) + 1/\epsilon V_s \bar{G}(x_s) \psi(x_s) ds \right]$$

and letting $t \rightarrow \infty$ yields by (4.17)

$$w_\epsilon(x) \leq J_x^\epsilon(V).$$

Lastly letting

$$\hat{V}_t = \begin{cases} 0 & w_\epsilon(x_t) < \bar{G}(x_t) \psi(x_t) \\ 1 & w_\epsilon(x_t) \geq \bar{G}(x_t) \psi(x_t) \end{cases}$$

it follows that

$$w_\epsilon(x) = J_x(\hat{V}) \quad \text{and we have uniqueness.}$$

Remark 5.2. Note that Lemmas 5.4 and 5.5 are valid for the case when $\bar{G}(x)\psi(x)$ is replaced by

$$(5.10) \quad \psi_\beta(x) = E_x \left[\int_0^\infty e^{-\beta s} \bar{G}(x_s) \psi(x_s) ds \right]$$

since

$$\psi_\beta(x) \leq \bar{G}(x) \|\psi\|.$$

Lemma 5.6. $u_\epsilon(x) \downarrow u(x)$ as $\epsilon \rightarrow 0$.

Proof. An argument due to Menaldi (see [3] or [4]) shows that for $\alpha > 0$

$$u_{\epsilon, \alpha} \downarrow \quad \text{as} \quad \epsilon \rightarrow 0.$$

Hence Lemma 5.5 yields $u_\epsilon \downarrow$ as $\epsilon \rightarrow 0$.

Define

$$\tau_\epsilon = \inf\{t: u_\epsilon(x_t) \geq \psi(x_s) \bar{G}(x_t)\}.$$

Lemma 5.5 implies that

$$u_\epsilon(x) = E_x \left[u_\epsilon(x_{t \wedge \tau_\epsilon}) + \int_0^{t \wedge \tau_\epsilon} \bar{G}(x_s) f(x_s) ds \right].$$

Since on $\tau_\epsilon = \infty$

$$u_\epsilon(x_t) < \bar{G}(x_t) \psi(x_t), \text{ we have}$$

$$\lim_{t \rightarrow \infty} u_\epsilon(x_{t \wedge \tau_\epsilon}) = 0.$$

Because of quasi-left continuity on $(\tau < \infty)$

$$\lim_{t \rightarrow \infty} u_\epsilon(x_{t \wedge \tau_\epsilon}) = u_\epsilon(x_{\tau_\epsilon}) \geq \bar{G}(x_{\tau_\epsilon}) \psi(x_{\tau_\epsilon})$$

and we have

$$u_\epsilon(x) \geq J_x(\tau_\epsilon) \geq u(x).$$

Let $\psi_\beta(x)$ be as in (5.10). Define $w_\epsilon(x)$ and $w(x)$ by substituting ψ_β in place of $\bar{G}\psi$ in the definitions of u_ϵ and u . The above proof again yields $w_\epsilon \geq w$.

Suppose $E_x(\tau) < \infty$ and define

$$V_t^\tau = \begin{cases} 0 & t < \tau \\ 1 & t \geq \tau \end{cases}$$

Then for w_ϵ and w

$$J_x^\epsilon(V^\tau) - J_x^\epsilon(\tau) = E_x \left[\int_\tau^\infty e^{-1/\epsilon(s-\tau)} (\bar{G}(x_s) f(x_s) + 1/\epsilon \psi_\beta(x_s)) ds - \psi_\beta(x_\tau) \right]$$

By the Markov property since $\psi_\beta \in D_A$, it follows that

$$\psi_\beta(x_{\tau+t})e^{-t/\varepsilon} - \int_0^t e^{-s/\varepsilon} (A\psi_\beta - 1/\varepsilon \psi_\beta)(x_{\tau+s})ds$$

is a martingale with respect to $G_s = F_{s+\tau}$. Hence

$$\begin{aligned} -E_x[\psi_\beta(x_\tau)] &= E_x \left[\int_0^\infty e^{-s/\varepsilon} (A\psi_\beta - 1/\varepsilon \psi_\beta)(x_{\tau+s})ds \right] \\ &= E_x \left[\int_0^\infty e^{-s-\tau/\varepsilon} (A\psi_\beta - 1/\varepsilon \psi_\beta)(x_s)ds \right] \end{aligned}$$

Hence

$$\begin{aligned} J_x^\varepsilon(V^\tau) - J_x^\varepsilon(\tau) &= E_x \left[\int_\tau^\infty e^{-s-\tau/\varepsilon} \bar{G}(x_s)f(x_s) - A\psi_\beta(x_s)ds \right] \\ &\leq \varepsilon \|f - A\psi_\beta\| \end{aligned}$$

For the same reasons as in Lemma 5.4

$$w(x) = \inf\{J_x(\tau) : E_x[\tau] < \infty\}$$

and so

$$w \leq w_\varepsilon \leq w + \varepsilon \|f - A\psi_\beta\|$$

showing $w_\varepsilon \downarrow w$.

Lastly since

$$\|w - u\| \leq \|\psi_\beta - \bar{G}\psi\|$$

and

$$\|w_\varepsilon - u_\varepsilon\| \leq \|\psi_\beta - \bar{G}\psi\|. \text{ It is a standard fact from semi-group theory}$$

that

$$\lim_{\alpha \rightarrow \infty} \|\psi_\beta - \bar{G}\psi\| = 0 \text{ and we have } u_\varepsilon \downarrow u \text{ as } \varepsilon \rightarrow 0.$$

Lemma 5.7. $u_\alpha \downarrow u$.

Proof. Let w_α be given by

$$w_\alpha(x) = \inf_{\tau} J_x^\alpha(\tau)$$

where

$$J_x^\alpha(\tau) = E_x \left[\int_0^\tau e^{-\alpha t} \bar{G}(x_t) f(x_t) dt + e^{-\alpha \tau} \psi_\beta(x_\tau) \right]$$

and $w_{\varepsilon, \alpha}(x)$ be defined as $u_{\varepsilon, \alpha}$ but with ψ_β replacing $\bar{G}\psi$. Then it follows as in Lemma 5.6 that $\|w_{\varepsilon, \alpha} - w_\alpha\| \leq \varepsilon \|f - A\psi_\beta\|$.

Hence letting w_ε and w be as in Lemma 5.6,

$$\begin{aligned} |w_\alpha(x) - w(x)| &\leq \|w_\alpha - w_{\alpha, \varepsilon}\| + |w_{\alpha, \varepsilon}(x) - w_\varepsilon(x)| + \|w_\varepsilon - w\| \\ &\leq 2 \|\psi_\beta - \bar{G}\psi\| + |w_\alpha(x) - w(x)| \end{aligned}$$

Letting $\alpha \rightarrow 0$ and then $\beta \rightarrow \infty$ shows $u_\alpha(x) \rightarrow u(x)$. That $u_\alpha(x) \uparrow u(x)$ follows since $J_x^\alpha(\tau) \uparrow J_x(\tau)$.

Proof of Theorem 5.3. That u is in the set of solution in (i) follows since Lemmas 5.4 and 5.7 show $u \in C_b(R^+)$ and letting $\alpha \rightarrow 0$ in Theorem 5.1 (i) yields that

$$u \leq T_t u + \int_0^t T_s(\bar{G}f) ds \quad u \leq \bar{G}\psi.$$

It is standard that if $h(x)$ is any other solution then

$$h(x) \leq J_x(\tau) \quad \text{for any stopping time } \tau. \quad \text{Hence } u \text{ is the maximal}$$

solution.

What remains to prove is that

$$\hat{\tau} = \inf\{t: u(x_t) = \bar{G}(x_t)\psi(x_t)\} \quad \text{is optimal.}$$

Let $B = \{x: u(x) = \psi(x)\bar{G}(x)\}$. If $x \in B$, $\hat{\tau} \equiv 0$ and there is nothing to prove.

If $x \notin B$, find δ so that

$$u(x) + \delta < \psi(x)\bar{G}(x).$$

Define

$$\tau_R = \inf\{t: |x_t - x| \geq R\}$$

and

$$\tau_\delta = \inf\{t: u(x_t) \geq \bar{G}(x_t)\psi(x_t) - \delta\}$$

Since $u_\epsilon \rightarrow u$ uniformly on compact sets choose ϵ_δ so for $\epsilon < \epsilon_\delta$

$$\sup_{|x-y| \leq R} |u_\epsilon(y) - u(y)| < \delta/2.$$

For $s \in [0, \tau_\delta \wedge \tau_R)$

$$u_\epsilon(x_s) \leq u(x_s) + \delta/2 \leq \bar{G}(x_s)\psi(x_s) - \delta/2$$

and so Lemma 5.5 says

$$u_\epsilon(x) = E_x \left[u_\epsilon(x_{\tau_\delta \wedge \tau_R}) + \int_0^{\tau_\delta \wedge \tau_R} \bar{G}(x_s) f(x_s) ds \right]$$

Letting $\epsilon \rightarrow 0$

$$u(x) = E_x \left[u(x_{\tau_\delta \wedge \tau_R}) + \int_0^{\tau_\delta \wedge \tau_R} \bar{G}(x_s) f(x_s) ds \right]$$

Again since $u=0$ at ∞ and we have quasi-left continuity, we can let $R \rightarrow \infty$ yielding

$$(5.11) \quad u(x) = E_x \left[u(x_{\tau_\delta}) + \int_0^{\tau_\delta} \bar{G}(x_s) f(x_s) ds \right].$$

Lastly note that $\tau_\delta \uparrow \hat{\tau}$. This is so since $\tau_\delta \uparrow \tilde{\tau}$ as $\delta \rightarrow 0$ and by quasi-left continuity

$$x_{\tau_\delta} \rightarrow x_{\tilde{\tau}}^- \quad \text{on } (\tilde{\tau} < \infty)$$

Thus $u(x_{\tau_\delta}) \geq \bar{G}(x_{\tau_\delta})\psi(x_{\tau_\delta})$ on $(\tilde{\tau} < \infty)$ and since $\tilde{\tau} \leq \hat{\tau}$, $\tilde{\tau} = \hat{\tau}$ on $(\tilde{\tau} < \infty)$.

Therefore $\tilde{\tau} = \hat{\tau}$ a.s. P_x . Therefore (5.11) becomes

$$u(x) = E_x \left[u(x_{\hat{\tau}}) + \int_0^{\hat{\tau}} \bar{G}(x_s) f(x_s) ds \right] = J_x(\hat{\tau}).$$

§6. The Long Run Average Cost Problem.

Let $V^\alpha(x)$ be as in Theorem 3.5. Recall that

$$(6.1) \quad V^\alpha(x) = \inf_{\tau} E_x \left[\int_0^{\tau} e^{-\alpha s} \bar{G}(x_s) f(x_s) + \alpha(V^\alpha(0) + c_0) G(x_s) ds \right. \\ \left. + e^{-\alpha \tau} (V^\alpha(0) + g(x_\tau) \bar{G}(x_\tau) + c_0 G(x_\tau)) \right]$$

Assume (4.8) and let

$$r(x) = \frac{AG(x)}{\bar{G}(x)}.$$

Since $x_t: t \geq 0$ and $G(x)$ are both non-decreasing, $r(x) \geq 0$. Define

$$(6.2) \quad u_\alpha(x) = \inf_{\tau} E_x \left[\int_0^{\tau} e^{-\alpha s} \bar{G}(x_s) (f(x_s) + c_0 r(x_s) - \alpha V^\alpha(0)) ds \right. \\ \left. + e^{-\alpha \tau} \bar{G}(x_\tau) g(x_\tau) \right]$$

By Dynkin's formula

$$G(x) = E_x \left[e^{-\alpha \tau} G(x_\tau) - \int_0^{\tau} e^{-\alpha s} (AG(x_s) - \alpha G(x_s)) ds \right]$$

and

$$(1 - e^{-\alpha \tau}) V^\alpha(0) = \int_0^{\tau} \alpha e^{-\alpha s} ds V^\alpha(0)$$

Thus

$$(6.3) \quad u_\alpha(x) = V^\alpha(x) + V^\alpha(0) - c_0 G(x)$$

and if $\hat{\tau}_\alpha$ is optimal for V^α as given by Theorem 3.5, then it is also optimal for $u_\alpha(x)$ in (6.2).

$$(6.4) \quad \bar{V} = \frac{E_0 \left[\int_0^{\infty} \bar{G}(x_s) (f(x_s) + c_0 r(x_s)) ds \right]}{E_0 \left[\int_0^{\infty} \bar{G}(x_s) ds \right]}$$

Note that

$$0 = E_0 \left[G(x_t) - \int_0^t AG(x_s) ds \right]$$

which yields by (4.17) by letting $t \rightarrow \infty$

$$1 = E_0 \left[\int_0^\infty AG(x_s) ds \right] = E_0 \left[\int_0^\infty \bar{G}(x_s) r(x_s) ds \right].$$

Hence

$$(6.5) \quad \bar{V} = \frac{E_0 \left[\int_0^\infty \bar{G}(x_s) f(x_s) ds \right] + c_0}{E_0 \left[\int_0^\infty \bar{G}(x_s) ds \right]}.$$

Lemma 6.1. $0 \leq \lim_{\alpha \rightarrow 0} \alpha V^\alpha(0) \leq \bar{V}.$

Proof. Let

$$\hat{H}(\alpha) = E_0 \left[\int_0^\infty \alpha e^{-\alpha s} G(x_s) ds \right] = E_0 \left[e^{-\alpha \sigma} \right] \quad \text{where } \sigma \text{ is given by}$$

(2.3). From (6.1), we obtain that

$$0 \leq V^\alpha(0) \leq E_0 \left[\int_0^\infty e^{-\alpha s} \bar{G}(x_s) f(x_s) ds \right] + (V^\alpha(0) + c_0) \hat{H}(\alpha)$$

and so

$$0 \leq V^\alpha(0) \leq \frac{E_0 \left[\int_0^\infty e^{-\alpha s} \bar{G}(x_s) f(x_s) ds \right] + c_0 \hat{H}(\alpha)}{1 - \hat{H}(\alpha)}$$

Since $\lim_{\alpha \rightarrow 0} \frac{1 - H(\alpha)}{\alpha} = E_0(\sigma) = E_0 \left[\int_0^\infty \bar{G}(x_s) ds \right],$

$$0 \leq \lim_{\alpha \rightarrow 0} \alpha V^\alpha(0) \leq \bar{V}.$$

Let

$$v_{\alpha}(x) = \inf_{\tau} E_x \left[\int_0^{\tau} e^{-\alpha s} \bar{G}(x_s) (f(x_s) - V^{\alpha}(0)) + \alpha c_0 G(x_s) ds \right] \\ + e^{-\alpha \tau} (g(x_{\tau}) \bar{G}(x_{\tau}) + c_0 G(x_{\tau}))$$

and

$$\bar{v}_0(x) = \frac{E_x \left[\int_0^{\infty} \bar{G}(x_s) (f(x_s) + c_0 r(x_s) - \bar{V}) ds \right]}{\bar{G}(x)}.$$

Suppose $\rho = \sup \bar{v}_0(x)$

$$(6.6) \quad \bar{V} + \rho r(x) - \lambda \geq \beta > 0$$

Theorem 6.1. Under (2.1) - (4.2), (4.8), (4.16), (4.17) and (6.6)

(i) $v_{\alpha} \rightarrow v$ uniformly on compact sets where

$$v(x) = \inf_{\tau} E_x \left[\int_0^{\tau} \bar{G}(x_s) (f(x_s) - \lambda) ds + g(x_{\tau}) \bar{G}(x_{\tau}) + c_0 G(x_{\tau}) \right]$$

(ii) $\hat{\tau} = \inf\{t: v(x_t) = \bar{G}(x_t)g(x_t) + c_0 G(x_t)\}$ is optimal,

(iii) $v(x)$ is the maximal element of the set of solutions h of $h \in C_b(\mathbb{R}^+)$

$$h \leq T_t h + \int_0^t T_s ((f - \lambda) \bar{G}) ds \quad h \leq \bar{G}g + c_0 G.$$

Moreover $v(0) = 0$.

$$(iv) \quad \lambda = \frac{E_0 \left[\int_0^{\hat{\tau}} \bar{G}(x_s) f(x_s) ds + \bar{G}(x_{\hat{\tau}}) g(x_{\hat{\tau}}) + c_0 G(x_{\hat{\tau}}) \right]}{E_0 \left[\int_0^{\hat{\tau}} \bar{G}(x_s) ds \right]}$$

and

$$\lambda = \inf_{\tau} \frac{E_0 \left[\int_0^{\tau} \bar{G}(x_s) f(x_s) ds + \bar{G}(x_{\tau}) g(x_{\tau}) + c_0 G(x_{\tau}) \right]}{E_0 \left[\int_0^{\tau} \bar{G}(x_s) ds \right]}$$

$$(v) \quad \lim_{\alpha \rightarrow 0} \alpha V^{\alpha}(0) = \lambda$$

$$(vi) \quad \lambda = \frac{E_0 \left[\int_0^{\hat{\tau} \wedge \sigma} f(x_s) ds + g(x_{\tau}) I(\hat{\tau} < \sigma) + c_0 I(\hat{\tau} \geq \sigma) \right]}{E_0[\hat{\tau} \wedge \sigma]}$$

and

$$\lambda = \inf_{\tau} \frac{E_0 \left[\int_0^{\tau \wedge \sigma} f(x_s) ds + g(x_{\tau}) I(\tau < \sigma) + c_0 I(\tau \geq \sigma) \right]}{E_0[\tau \wedge \sigma]}.$$

Proof. By Theorem 3.5 and (6.3)

$$v_{\alpha}(x) = V^{\alpha}(x) - V^{\alpha}(0) = u_{\alpha}(x) + c_0 G(x)$$

and

$$v(x) = u(x) + c_0 G(x)$$

where

$$u(x) = \inf_{\tau} E_x \left[\int_0^{\tau} \bar{G}(x_s) (f(x_s) + c_0 r(x) - \lambda) ds + \bar{G}(x_{\tau}) g(x_{\tau}) \right]$$

Let

$$\bar{v}_0(x) = \frac{E_x \left[\int_0^{\infty} \bar{G}(x_s) (f(x_s) + c_0 r(x_s) - \bar{V}) ds \right]}{\bar{G}(x)}$$

and $\bar{v}(x) = \bar{v}_0(x) - \rho$. Note that $\bar{v}(x) \leq 0$ and $\bar{v}(0) = -\rho \leq 0$. Define

$$w_{\alpha}(x) = u_{\alpha}(x) - \bar{G}(x) \bar{v}(x) \quad \text{and}$$

$$w(x) = u(x) - \bar{G}(x) \bar{v}(x).$$

By Theorems 4.1 and 4.4, $\bar{v}(x)$ is the unique solution of $-\bar{A}\bar{v} = f(x) + c_0 r(x) - \bar{V}$ with $\bar{v}(0) = -\rho$ and

$$-A\bar{v} = \frac{A(\bar{G}\bar{v})}{\bar{G}} + \bar{v}(0) \frac{AG}{\bar{G}}.$$

So

$$-A(\bar{G}\bar{v})(x) = \bar{G}(x)(f(x) + c_0 r(x) - \bar{V}) + v(0)AG(x)$$

Therefore

$$e^{-\alpha t} \bar{G}(x_t) \bar{v}(x_t) + \int_0^t e^{-\alpha s} \bar{G}(x_s) (f(x_s) + c_0 r(x_s) - \bar{V}) + \bar{v}(0) AG(x_s) - \alpha \bar{G}(x_s) \bar{v}(x_s) ds$$

and

$$\bar{G}(x_t) \bar{v}(x_t) + \int_0^t \bar{G}(x_s) (f(x_s) + c_0 r(x_s) - \bar{V}) + \bar{v}(0) AG(x_s) ds$$

are both martingales. Hence

$$w_\alpha(x) = \inf_{\tau} E_x \left[\int_0^{\tau} e^{-\alpha s} \bar{G}(x_s) (\bar{V} + \rho r(x_s) - \alpha v^\alpha(0)) ds + e^{-\alpha \tau} \bar{G}(x_\tau) (g(x_\tau) - \bar{v}(x_\tau)) \right]$$

By (4.17), $\lim_{t \rightarrow \infty} \bar{G}(x_t) = 0$ and by (2.2), $x_t: t \geq 0$ is quasi-left continuous and so

for all stopping times τ ,

$$\begin{aligned} E_x \left[\int_0^{\tau} \bar{G}(x_s) (f(x_s) + c_0 r(x_s) - \lambda) ds + \bar{G}(x_\tau) g(x_\tau) - \bar{G}(x) \bar{v}(x) \right] \\ = E_x \left[\int_0^{\tau} \bar{G}(x_s) (\bar{V} + \rho r(x_s) - \lambda) ds + \bar{G}(x_\tau) (g(x_\tau) - \bar{v}(x_\tau)) \right] \end{aligned}$$

and so

$$w(x) = \inf_{\tau} E_x \left[\int_0^{\tau} \bar{G}(x_s) (\bar{V} + \rho r(x_s) - \lambda) ds + \bar{G}(x_\tau) (g(x_\tau) - \bar{v}(x_\tau)) \right]$$

Define

$$\tilde{w}_\alpha(x) = \inf_{\tau} E_x \left[\int_0^{\tau} e^{-\alpha s} \bar{G}(x_s) (\bar{V} + \rho r(x_s) - \lambda) ds + e^{-\alpha \tau} \bar{G}(x_\tau) (g(x_\tau) - \bar{v}(x_\tau)) \right]$$

Let $\alpha_n \rightarrow 0$ so that by Lemma 6.1,

$$\lim_n \alpha_n V^{\alpha_n}(0) = \lambda$$

By Theorem 5.3

$$\tilde{w}_{\alpha_n}(x) \rightarrow w(x) \text{ uniformly on compact sets.}$$

Now

$$|w_{\alpha_n}(x) - \tilde{w}_{\alpha_n}(x)| \leq (|\lambda - \alpha_n V^{\alpha_n}(0)| + \alpha_n \|\bar{v}\|) E_x \left[\int_0^\infty \bar{G}(x_s) ds \right]$$

Since (4.16) say

$$E_x \left[\int_0^\infty \bar{G}(x_s) ds \right] \leq C \text{ independent of } x$$

$$\lim_{n \rightarrow \infty} \|w_{\alpha_n} - \tilde{w}_{\alpha_n}\| = 0$$

Also

$$\begin{aligned} |v_{\alpha_n}(x) - v(x)| &\leq |w_{\alpha_n}(x) - w(x)| \\ &\leq \|\tilde{w}_{\alpha_n} - w_{\alpha_n}\| + |\tilde{w}_{\alpha_n}(x) - w(x)| \end{aligned}$$

and so (i) follows for the sequence $\{\alpha_n\}^\infty$.

From Theorem 5.3

$$\hat{\tau} = \inf\{t: w(x_t) = \bar{G}(x_t)(g(x_t) - \bar{v}(x_t))\}$$

is optimal. It follows that

$$\hat{\tau} = \inf\{t: v(x_t) = \bar{G}(x_t)g(x_t) + c_0 G(x_t)\}$$

is optimal for v . All we need notice is that

$$v(x) = w(x) + \bar{G}(x)\bar{v}(x) + c_0 G(x).$$

Since $v_\alpha(0) = 0$, it follows that $v(0) = 0$ and

$$\lambda = \inf_{\tau} \frac{E_x \left[\int_0^{\tau} \bar{G}(x_s) f(x_s) ds + \bar{G}(x_{\tau}) g(x_{\tau}) + c_0 G(x_{\tau}) \right]}{E_0 \left[\int_0^{\tau} \bar{G}(x_s) ds \right]}$$

Thus if for any other subsequence $\alpha_n \rightarrow 0$ with $\alpha_n V^{\alpha_n}(0) \rightarrow \bar{\lambda}$, then repeating the above argument we see that $\bar{\lambda} = \lambda$. Hence (i) and (v) follow. Now (ii) and (iv) are also done and (v) follows from Lemmas 3.2 and 3.3.

For (iii), we have from Theorem 5.4 that w is the maximal solution of

$$\ell \in C_b(R^+) \quad \ell \leq T_t \ell + \int_0^t T_s (\bar{G}(\bar{V} + \rho r - \lambda)) ds \quad \ell \leq \bar{G}(g - \bar{v})$$

Since

$$\bar{G} \bar{v} = T_t \bar{G} \bar{v} - \int_0^t T_s (\bar{G}(f + c_0 r - \bar{V}) + \rho A G) ds$$

and

$$\begin{aligned} G &= T_t G - \int_0^t T_s (AG) ds \\ &= T_t G - \int_0^t T_s (\bar{G}r) ds \end{aligned}$$

we see that

$$\begin{aligned} v(x) &= w(x) + \bar{G}(x) \bar{v}(x) + c_0 G(x) \\ &\leq T_t w + \int_0^t T_s (\bar{G}(\bar{V} + \rho r - \lambda)) ds \\ &\quad + T_t \bar{G} \bar{v} - \int_0^t T_s (\bar{G}(f + c_0 r - \bar{V} - \rho r)) ds \\ &\quad + c_0 T_t G - \int_0^t T_s (\bar{G} - c_0 r) ds \\ &= T_t v + \int_0^t T_s (\bar{G}(f - \lambda)) ds \end{aligned}$$

and

$$v \leq \bar{G}g + c_0 G.$$

Now if h is any solution of (iv), then

$$h(x_t) + \int_0^t \bar{G}(x_s)(f(x_s) - \lambda)ds \text{ is a submartingale and so for any}$$

stopping time τ ,

$$\begin{aligned} h(x) &\leq E_x \left[h(x_{t \wedge \tau}) + \int_0^{t \wedge \tau} \bar{G}(x_s)(f(x_s) - \lambda)ds \right] \\ &\leq E_x \left[\int_0^{t \wedge \tau} \bar{G}(x_s)(f(x_s) - \lambda)ds + \bar{G}(x_{t \wedge \tau})g(x_{t \wedge \tau}) + c_0 G(x_{t \wedge \tau}) \right] \end{aligned}$$

By (4.17) and (2.2) letting $t \rightarrow \infty$

$$h(x) \leq E_x \left[\int_0^\tau \bar{G}(x_s)(f(x_s) - \lambda)ds + \bar{G}(x_\tau)g(x_\tau) + c_0 G(x_\tau) \right].$$

Since $\hat{\tau}$ is optimal

$$h(x) \leq v(x).$$

Remark 6.1. Recall that

$$\hat{\tau} = \inf\{t: v(x_t) = \bar{G}(x_t)g(x_t) + c_0 G(x_t)\}.$$

Under the assumption $g(0) > 0$ since $v(0) = 0$ it follows that

$$P_0(\hat{\tau} > 0) = 1.$$

Remark 6.2. Since $\bar{v}_0(0) = 0$ it follows from Theorems 4.1 and 4.4 that for any stopping time τ ,

$$0 = E_0 \left[\bar{G}(x_\tau)\bar{v}_0(x_\tau) + \int_0^\tau \bar{G}(x_s)(f(x_s) + c_0 r(x_s) - \bar{v})ds \right]$$

Theorem 6.2. Suppose for $U \subset \mathbb{R}^+$ open and not containing a neighborhood of 0 that if $\tau = \inf\{t: x(t) \in U\}$, then $P_0(\tau < \infty) > 0$. Assume further that $G(x) < 1$ for all x . Under the conditions (2.1) - (2.4), (4.8) (4.16), (4.17) and $g(0) > 0$, if $\lambda < \bar{V}$ then $\{g(x) < \bar{v}_0(x)\} \neq \emptyset$. If $r(x) > \beta > 0$ and $\{g(x) < \bar{v}_0(x)\} \neq \emptyset$ then $\lambda < \bar{V}$.

Proof. Suppose $\lambda < \bar{V}$ and let $\hat{\tau}$ be optimal. Note that

$$(6.7) \quad P_0(\hat{\tau} < \infty) > 0.$$

Now

$$(6.8) \quad 0 = E_0 \left[\int_0^{\hat{\tau}} \bar{G}(x_s) (f(x_s) + c_0 r(x_s) - \lambda) ds + \bar{G}(x_{\hat{\tau}}) g(x_{\hat{\tau}}) \right]$$

because for any stopping time τ ,

$$E_0[G(x_\tau)] = E_0 \left[\int_0^\tau \bar{G}(x_s) r(x_s) ds \right].$$

So by Remark 6.2 and (6.8)

$$0 = E_0 \left[\int_0^{\hat{\tau}} \bar{G}(x_s) (\bar{V} - \lambda) ds + \bar{G}(x_{\hat{\tau}}) (g(x_{\hat{\tau}}) - \bar{v}_0(x_{\hat{\tau}})) \right]$$

and by Remark 6.1 and (6.7)

$$P_0(0 < \hat{\tau} < \infty) > 0.$$

Hence

$$\{g(x) < \bar{v}_0(x)\} \neq \emptyset.$$

Suppose $r(x) > \beta > 0$ and $\{g(x) < \bar{v}_0(x)\} \neq \emptyset$. Since both g and $\bar{v}_0(x)$ are continuous, for $\delta > 0$ but small enough

$$U = \{x(x) + \delta < \bar{v}_0(x)\} \text{ and does not contain a neighborhood of the}$$

origin. Define

$\tau = \inf\{t: g(x_t) + \delta < \bar{v}_0(x_t)\}$ and note then that $P_0(0 < \tau < \infty) > 0$. Also since $g(x) \geq 0$ and $\{\bar{v}_0(x) > g(x)\} \neq \emptyset$ then $\rho = \sup \bar{v}_0(x) > 0$ and Theorem 6.1 applies. Arguing as above

$$0 \leq E_0 \left[\int_0^\tau \bar{G}(x_s) (\bar{V} - \lambda) ds + G(x_\tau) (g(x_\tau) - v_0(x_\tau)) \right]$$

and so

$$0 < \delta E_0[\bar{G}(x_\tau)] \leq E_0 \left[\int_0^\tau \bar{G}(x_s) (\bar{V} - \lambda) ds \right], \text{ that is } \lambda < \bar{V}.$$

Theorem 6.3.

(i) Under the conditions (2.1)-(2.4), (4.8), (4.16), (4.17) and $g(x) > 0$, if $\lambda = \bar{V}$ and $r(x) \geq \beta > 0$ then the do nothing policy is optimal.

(ii) Under the conditions (2.1)-(2.4), (4.8), (4.17) and if $\rho = \sup \bar{v}_0(x) = 0$ and $\lambda = \bar{V}$ then the do nothing policy is optimal.

Proof. For (i), note that Theorem 6.1 applies and since

$$\lambda = \bar{V} = \frac{E_0 \left[\int_0^\infty \bar{G}(x_s) f(x_s) ds \right] + c_0}{E_0 \left[\int_0^\infty \bar{G}(x_s) ds \right]}$$

the do nothing policy is optimal.

For (ii) if $\rho = \sup \bar{v}_0(x) = 0$ we have for any stopping time τ

$$\begin{aligned} E_0 \left[\int_0^\tau \bar{G}(x_s) (f(x_s) - \lambda) ds + \bar{G}(x_\tau) g(x_\tau) + c_0 G(x_\tau) \right] \\ = E_0 \left[\int_0^\tau \bar{G}(x_s) (\bar{V} - \lambda) ds + \bar{G}(x_\tau) (g(x_\tau) - \bar{v}_0(x_\tau)) \right] \geq 0 \end{aligned}$$

and since

$$E_0 \left[\int_0^{\infty} \bar{G}(x_s) (f(x_s) - \lambda) ds \right] + c_o = 0$$

again the do nothing policy is optimal.

References.

- [1]. Abdel-Hameed, M., Life Distributions Properties of Devices Subject to Pure Jump Damage Processes, J. App. Prob., 21 (1984), pp. 816 - 825.
- [2]. Abdel-Hameed, M., Life Distribution Properties of Devices Subject to Lévy Wear Processes, Math. Op. Res., 9 (1984), pp. 606-614.
- [3]. Bensoussan, A., Stochastic Control by Functional Analysis Methods, North-Holland, Amsterdam, 1982.
- [4]. Bensoussan, A. and Lions, J. L., Impulse Control and Quasi-Variational Inequalities, Gauthier-Villars, Paris, 1984.
- [5]. Drosen, J. W., Failure Times and Optimal Stopping Rules of Generalized Shock Models, Thesis, Northwestern University.
- [6]. Dynkin, E. B., Markov Processes, Springer-Verlag, New York, 1965.
- [7]. Feller, W., An Introduction to Probability Theory and its Applications, Wiley, New York, 1966.
- [8]. Robin, M., On Some Impulse Control Problems with Long Run Average Cost, SIAM J. Control and Optimization, 19 (1981), pp. 333 - 358.
- [9]. Robin, M., Contrôle Impulsionnel des Processus de Markov, Thèse d'état, Université de Paris IX, 1976.
- [10]. Taylor, H., Optimal Replacement under Additive Damage and other Failure Models, Navel Res. Log. Q., 22 (1975), pp. 1 - 18.

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